PROBLEMS ON ANALYSIS

We use the notation \( f(x) \sim g(x) \) to mean \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1 \). One says that \( f(x) \) is asymptotic to \( g(x) \).

1. Show that \( \int_0^\infty \frac{\cos(ax)}{1+x^2} \, dx \) exists for \( a \in \mathbb{R} \) and compute its value.

2. Find a simple function \( f(x) \) for which \( x^{1/x} - 1 \sim f(x) \) as \( x \to \infty \).

3. For what pairs \( (a, b) \) of positive real numbers does the improper integral

\[
\int_b^\infty \left( \sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) \, dx
\]

converge?

4. Let \( a_n \) be the unique positive root of \( x^n + x = 1 \). Find a simple function \( f(n) \) for which \( 1 - a_n \sim f(n) \) as \( n \to \infty \).

5. For each continuous function \( f : [0, 1] \to \mathbb{R} \), let \( I(f) = \int_0^1 x^2 f(x) \, dx \) and \( J(f) = \int_0^1 x f(x)^2 \, dx \). Find the maximum value of \( I(f) - J(f) \) over all such functions \( f \).

6. For a positive real number \( a \), calculate \( \int_0^\infty t^{-1/2} e^{-a(t+t^{-1})} \, dt \).

7. Let \( f \) be a function on \([0, \infty)\), differentiable and satisfying

\[
f'(x) = -3f(x) + 6f(2x)
\]

for \( x > 0 \). Assume that \( |f(x)| \leq e^{-\sqrt{x}} \) for \( x \geq 0 \) (so that \( f(x) \) tends rapidly to \( \infty \) as \( x \) increases). For \( n \) a nonnegative integer, define

\[
\mu_n = \int_0^\infty x^n f(x) \, dx
\]

(the \( n \)th moment of \( f \)).

(a) Express \( \mu_n \) in terms of \( \mu_0 \).

(b) Prove that the sequence \( \{ \mu_n \cdot 3^n/n! \} \) always converges, and that the limit is 0 only if \( \mu_0 = 0 \).

8. Suppose \( f \) and \( g \) are non-constant, differentiable, real-valued functions defined on \((-\infty, \infty)\). Furthermore, suppose that for each pair of real numbers \( x \) and \( y \),

\[
\begin{align*}
f(x + y) &= f(x)f(y) - g(x)g(y), \\
g(x + y) &= f(x)g(y) + g(x)f(y).
\end{align*}
\]

If \( f'(0) = 0 \), prove that \( (f(x))^2 + (g(x))^2 = 1 \) for all \( x \).

9. Let \( a \) and \( b \) be positive numbers. Find the largest number \( c \), in terms of \( a \) and \( b \), such that

\[
a^x b^{1-x} \leq a \frac{\sinh ux}{\sinh u} + b \frac{\sinh u(1-x)}{\sinh u}
\]

for all \( u \) with \( 0 < |u| \leq c \) and for all \( x, 0 < x < 1 \). (Note: \( \sinh u = (e^u - e^{-u})/2 \).)
10. The function $K(x, y)$ is positive and continuous for $0 \leq x \leq 1, 0 \leq y \leq 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \leq x \leq 1$. Suppose that for all $x, 0 \leq x \leq 1$,

$$\int_0^1 f(y)K(x, y) \, dy = g(x)$$

and

$$\int_0^1 g(y)K(x, y) \, dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \leq x \leq 1$.

11. Evaluate

$$\int_0^\infty \left( x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right) \left( 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right) \, dx.$$

12. Let $f$ be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real $x$. Prove that $|f(x)|$ is bounded.

13. Prove that there is a constant $C$ such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \leq C \int_{-1}^1 |p(x)| \, dx.$$

14. Find a real number $c$ and a positive number $L$ for which

$$\lim_{r \to \infty} r^c \int_0^{\pi/2} x^r \sin x \, dx = L.$$

15. Let $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers $x$ and $y$ such that

$$(a_1, b_1)x^{a_1}y^{b_1} + (a_2, b_2)x^{a_2}y^{b_2} + \cdots + (a_n, b_n)x^{a_n}y^{b_n} = (0, 0).$$

16. Show that all solutions of the differential equation $y'' + 2xy = 0$ remain bounded as $x \to \infty$.

17. Let $f$ be a real-valued function having partial derivatives and which is defined for $x^2 + y^2 \leq 1$ and is such that $|f(x, y)| \leq 1$. Show that there exists a point $(x_0, y_0)$ in the interior of the unit circle for which

$$\left( \frac{\partial f}{\partial x}(x_0, y_0) \right)^2 + \left( \frac{\partial f}{\partial y}(x_0, y_0) \right)^2 \leq 16.$$ 

18. (a) On $[0, 1]$, let $f$ have a continuous derivative satisfying $0 < f'(x) \leq 1$. Also, suppose that $f(0) = 0$. Prove that

$$\left( \int_0^1 f(x) \, dx \right)^2 \geq \int_0^1 f(x)^3 \, dx.$$

(b) Find an example where equality occurs.
19. Let \( P(t) \) be a nonconstant polynomial with real coefficients. Prove that the system of simultaneous equations
\[
0 = \int_0^x P(t) \sin t\,dt = \int_0^x P(t) \cos t\,dt
\]
has only finitely many real solutions \( x \).

20. Let \( C \) be the class of all real valued continuously differentiable functions \( f \) on the interval \( 0 \leq x \leq 1 \) with \( f(0) = 0 \) and \( f(1) = 1 \). Determine the largest real number \( u \) such that
\[
u \leq \int_0^1 |f'(x) - f(x)|\,dx
\]
for all \( f \in C \).

21. Given a convergent series \( \sum a_n \) of positive terms, prove that the series \( \sum \sqrt[n]{a_1a_2\cdots a_n} \) must also be convergent.

22. Given that \( f(x) + f'(x) \to 0 \) as \( x \to \infty \), prove that both \( f(x) \to 0 \) and \( f'(x) \to 0 \).

23. Suppose that \( f''(x) \) is continuous on \( \mathbb{R} \), and that \( |f'(x)| \leq a \) on \( \mathbb{R} \), and \( |f''(x)| \leq b \) on \( \mathbb{R} \). Find the best possible bound \( |f'(x)| \leq c \) on \( \mathbb{R} \).

24. Let \( f \) be a real function with a continuous third derivative such that \( f(x), f'(x), f''(x), f'''(x) \) are positive for all \( x \). Suppose that \( f'''(x) \leq f(x) \) for all \( x \). Show that \( f'(x) < 2f(x) \) for all \( x \). (Note that we cannot replace 2 by 1 because of the function \( f(x) = e^x \).)

25. Show that the improper integral
\[
\lim_{B \to \infty} \int_0^B \sin(x) \sin(x^2)\,dx
\]
converges.

26. Fix an integer \( b \geq 2 \). Let \( f(1) = 1, f(2) = 2 \), and for each \( n \geq 3 \), define \( f(n) = nf(d) \), where \( d \) is the number of base-\( b \) digits of \( n \). For which values of \( b \) does
\[
\sum_{n=1}^{\infty} \frac{1}{f(n)}
\]
converge?

27. Evaluate
\[
\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left( \frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.
\]

28. Find all differentiable functions \( f : (0, \infty) \to (0, \infty) \) for which there is a positive real number \( a \) such that
\[
f'(\frac{a}{x}) = \frac{x}{f(x)}
\]
for all \( x > 0 \).
29. Let $k$ be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[4]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n \to \infty} \frac{a_{n+1}}{n^k}.$$

30. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ xf(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

31. Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that for every rational number $q$, the number $f(q)$ is rational and has the same denominator as $q$. (The denominator of a rational number $q$ is the unique positive integer $b$ such that $q = a/b$ for some integer $a$ with $\gcd(a, b) = 1$.) (Note: gcd means greatest common divisor.)

32. Functions $f, g, h$ are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$f' = 2f^2gh + \frac{1}{gh}, \quad f(0) = 1,$$
$$g' = fg^2h + \frac{4}{fh}, \quad g(0) = 1,$$
$$h' = 3fh^2 + \frac{1}{fg}, \quad h(0) = 1.$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

33. Let $f : [0, 1]^2 \to \mathbb{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0,1)^2$. Let $a = \int_0^1 f(0, y) \, dy$, $b = \int_0^1 f(1, y) \, dy$, $c = \int_0^1 f(x, 0) \, dx$, $d = \int_0^1 f(x, 1) \, dx$. Prove or disprove: There must be a point $(x_0, y_0)$ in $(0,1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0, y_0) = b - a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0) = d - c.$$
36. Suppose that the function \( h : \mathbb{R}^2 \to \mathbb{R} \) has continuous partial derivatives and satisfies the equation
\[
h(x, y) = a \frac{\partial h}{\partial x}(x, y) + b \frac{\partial h}{\partial y}(x, y)
\]
for some constants \( a, b \). Prove that if there is a constant \( M \) such that \( |h(x, y)| \leq M \) for all \((x, y) \in \mathbb{R}^2\), then \( h \) is identically zero.

37. Let \( f : [0, \infty) \to \mathbb{R} \) be a strictly decreasing continuous function such that \( \lim_{x \to \infty} f(x) = 0 \). Prove that \( \int_0^\infty \frac{f(x) - f(x+1)}{f(x)} \, dx \) diverges.

38. Is there a strictly increasing function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f'(x) = f(f(x)) \) for all \( x \)?