Problem 1 Let $A$ be a $2 \times 2$ real matrix and consider the linear system of first order differential equations,

$$y'(t) = Ay(t), \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.$$ 

Let $\alpha$ be a real number, let $\beta$ be a nonzero real number, and let $M_1, M_2$ be $2 \times 2$ matrices with real entries. Suppose that the general solution of the linear system is,

$$y(t) = (k_1 M_1 + k_2 M_2) \begin{bmatrix} e^{\alpha t} \cos(\beta t) \\ e^{\alpha t} \sin(\beta t) \end{bmatrix},$$

where $k_1, k_2$ are arbitrary real numbers.

(a) Prove that $M_1$ and $M_2$ each satisfy the following equation,

$$AM_i = M_i D, \quad D = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$$ 

Solution: By assumption,

$$AM_i \begin{bmatrix} e^{\alpha t} \cos(\beta t) \\ e^{\alpha t} \sin(\beta t) \end{bmatrix} = M_i \frac{d}{dt} \begin{bmatrix} e^{\alpha t} \cos(\beta t) \\ e^{\alpha t} \sin(\beta t) \end{bmatrix}. $$

And,

$$\frac{d}{dt} \begin{bmatrix} e^{\alpha t} \cos(\beta t) \\ e^{\alpha t} \sin(\beta t) \end{bmatrix} = \begin{bmatrix} \alpha e^{\alpha t} \cos(\beta t) - \beta e^{\alpha t} \sin(\beta t) \\ \alpha e^{\alpha t} \sin(\beta t) + \beta e^{\alpha t} \sin(\beta t) \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} e^{\alpha t} \cos(\beta t) \\ e^{\alpha t} \sin(\beta t) \end{bmatrix}. $$

Therefore, for each real number $t$,

$$(AM_i - M_i D) \begin{bmatrix} e^{\alpha t} \cos(\beta t) \\ e^{\alpha t} \sin(\beta t) \end{bmatrix} = 0.$$ 

But for $t = 0$ and $t = \pi/(2\beta)$, the vectors give a basis for $\mathbb{R}^2$. Therefore $AM_i - M_i D = 0$.

(b) Consider the linear system of differential equations,

$$z'(t) = A^2 z(t), \quad z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}. $$

Use (a) to show that for every pair of real numbers $k_1, k_2$, the following function is a solution of the linear system,

$$z(t) = (k_1 M_1 + k_2 M_2) \begin{bmatrix} e^{(\alpha^2 - \beta^2)t} \cos(2\alpha \beta t) \\ e^{(\alpha^2 - \beta^2)t} \sin(2\alpha \beta t) \end{bmatrix}. $$

Solution: Because $AM_i = M_i D$, also

$$A^2 M_i = A(AM_i) = A(M_i D) = (AM_i) D = (M_i D) D = M_i D^2.$$ 

Now,

$$D^2 = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} \alpha^2 - \beta^2 & -2\alpha \beta \\ 2\alpha \beta & \alpha^2 - \beta^2 \end{bmatrix}. $$
And,
\[
\frac{d}{dt} \left[ e^{(a^2-b^2)t} \cos(2a\beta t) \right] = \left[ \begin{array}{cc} a^2 - b^2 & -2a\beta \\ 2a\beta & a^2 - b^2 \end{array} \right] e^{(a^2-b^2)t} \cos(2a\beta t)
\]
Thus,
\[
\frac{d}{dt} M_d \left[ e^{(a^2-b^2)t} \cos(2a\beta t) \right] = M_d D^2 \left[ e^{(a^2-b^2)t} \cos(2a\beta t) \right] = A^2 M_i \left[ e^{(a^2-b^2)t} \cos(2a\beta t) \right].
\]
Therefore, for each pair of real numbers \( k_1, k_2, \)
\[
\frac{d}{dt} (k_1 M_1 + k_2 M_2) \left[ e^{(a^2-b^2)t} \cos(2a\beta t) \right] = A^2 (k_1 M_1 + k_2 M_2) \left[ e^{(a^2-b^2)t} \cos(2a\beta t) \right],
\]
i.e.,
\[
z(t) = (k_1 M_1 + k_2 M_2) \left[ e^{(a^2-b^2)t} \cos(2a\beta t) \right],
\]
is a solution of \( z'(t) = A^2 z(t). \)

**Problem 2** Consider the following inhomogeneous 2nd order linear differential equation,
\[
\begin{cases}
y'' - y = 1, \\
y(0) = y_0, \\
y'(0) = v_0
\end{cases}
\]
Denote by \( Y(s) \) the Laplace transform,
\[
Y(s) = \mathcal{L}[y(t)] = \int_0^\infty e^{-st} y(t) \, dt.
\]
(a) Find an expression for \( Y(s) \) as a sum of ratios of polynomials in \( s \).

**Solution:** By rules of the Laplace transform, \( \mathcal{L}[y'(t)] = sY(s) - y_0 \) and \( \mathcal{L}[y''(t)] = s^2Y(s) - sy_0 - v_0 \). Therefore,
\[
(s^2Y(s) - sy_0 - v_0) - Y(s) = \mathcal{L}[y'' - y] = \mathcal{L}[y] = \frac{1}{s}.
\]
Gathering terms and simplifying,
\[
(s - 1)(s + 1)Y(s) = (s^2 - 1)Y(s) = v_0 + sy_0 + \frac{1}{s}.
\]
Therefore,
\[
Y(s) = \frac{s^2y_0 + sv_0 + 1}{(s + 1)s(s - 1)}.
\]
(b) Determine the partial fraction expansion of \( Y(s) \).

**Solution:** Because each factor in the denominator is a linear factor with multiplicity 1, the Heaviside cover-up method determines all the coefficients,
\[
\frac{s^2y_0 + sv_0 + 1}{(s + 1)s(s - 1)} = \frac{y_0 - v_0 + 1}{2} \frac{1}{s + 1} + (-1) \frac{1}{s} + \frac{y_0 + v_0 + 1}{2} \frac{1}{s - 1}.
\]
(c) Determine \( y(t) \) by computing the inverse Laplace transform of \( Y(s) \).

**Solution:** The inverse Laplace transform of \( 1/(s - a) \) is \( e^{at} \). Therefore,
\[
y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{y_0 - v_0 + 1}{2} e^{-t} - 1 + \frac{y_0 + v_0 + 1}{2} e^t = -1 + (y_0 + 1) \cosh(t) + v_0 \sinh(t).
\]
Problem 3 The general skew-symmetric real $2 \times 2$ matrix is,

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix},$$

where $b$ is a real number. Prove that the eigenvalues of $A$ of the form $\lambda = \pm i\mu$ for some real number $\mu$. Determine $\mu$ and find all values of $b$ such that there is a single repeated eigenvalue.

Solution: The trace is $\text{Trace}(A) = 0$, and the determinant is $\det(A) = 0 - (-b^2) = b^2$. Therefore the characteristic polynomial is

$$p_A(\lambda) = \lambda^2 - \text{Trace}(A)\lambda + \det(A) = \lambda^2 + b^2.$$ 

Therefore the eigenvalues of $A$ are $\pm ib$. There is a repeated eigenvalue iff $b = 0$.

There is a more involved proof that for every positive integer $n$, for every skew-symmetric real $n \times n$ matrix $A$, every eigenvalue of $A$ is purely imaginary. The idea is that on $\mathbb{C}^n$ there is a Hermitian inner product, which assigns to each pair of vectors, $z \in \mathbb{C}^n$, $w \in \mathbb{C}^n$, the complex number,

$$\langle z, w \rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}.$$ 

Observe this has the properties,

$$\langle z_1 + z_2, w \rangle = \langle z_1, w \rangle + \langle z_2, w \rangle,$$

$$\langle z, w_1 + w_2 \rangle = \langle z, w_1 \rangle + \langle z, w_2 \rangle,$$

$$\langle \lambda z, w \rangle = \lambda \langle z, w \rangle,$$

$$\langle z, \lambda w \rangle = \overline{\lambda} \langle z, w \rangle,$$

$$\langle w, z \rangle = \overline{\langle z, w \rangle},$$

$$\langle z, z \rangle \neq 0, \text{ if } z \neq 0.$$ 

Because $A$ is a real skew-symmetric matrix, for every pair of vectors $z, w$ the following equation holds,

$$\langle Az, w \rangle = -\langle z, Aw \rangle.$$ 

Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue and let $z$ be a (nonzero) $\lambda$-eigenvalue. Then,

$$\lambda \langle z, z \rangle = \langle \lambda z, z \rangle = \langle Az, z \rangle = -\langle z, Az \rangle = -\langle z, \lambda z \rangle = -\overline{\lambda} \langle z, z \rangle.$$ 

Because $z$ is nonzero, $\langle z, z \rangle$ is nonzero. Therefore $\lambda = -\overline{\lambda}$, which implies that $\lambda$ is a pure imaginary number.

Problem 4 Let $\lambda$ be a real number and let $A$ be the following $3 \times 3$ matrix,

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$ 

Let $a_1, a_2, a_3$ be real numbers. Consider the following initial value problem,

$$\begin{cases}
    y'(t) = Ay(t), \\
    y(0) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.
\end{cases}$$
Denote by $Y(s)$ the Laplace transform of $y(t)$, i.e.,

$$
Y(s) = \begin{bmatrix}
Y_1(s) \\
Y_2(s) \\
Y_3(s)
\end{bmatrix}, \quad Y_i(s) = \mathcal{L}[y_i(t)], \; i = 1, 2, 3.
$$

(a) Express both $\mathcal{L}[y'(t)]$ and $\mathcal{L}[Ay(t)]$ in terms of $Y(s)$.

**Solution:** First of all,

$$
\mathcal{L}[y'(t)] = sY(s) - \begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}.
$$

Secondly,

$$
\mathcal{L}[Ay(t)] = AY(s).
$$

(b) Using part (a), find an equation that $Y(s)$ satisfies, and iteratively solve the equation for $Y_3(s)$, $Y_2(s)$ and $Y_1(s)$, in that order.

**Solution:** By part (a),

$$
sY(s) - \begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix} = AY(s).
$$

Written out, this is equivalent to the system of 3 equations,

$$
\begin{align*}
(s - \lambda)Y_1(s) &= a_1 + Y_2(s) \\
(s - \lambda)Y_2(s) &= a_2 + Y_3(s) \\
(s - \lambda)Y_3(s) &= a_3
\end{align*}
$$

Solving this iteratively,

$$
Y_3(s) = \frac{a_3}{s - \lambda},
$$

$$
Y_2(s) = \frac{a_2}{s - \lambda} + \frac{1}{(s - \lambda)^2}Y_3(s) = \frac{a_2}{s - \lambda} + \frac{a_3}{(s - \lambda)^2},
$$

and,

$$
Y_1(s) = \frac{a_1}{s - \lambda} + \frac{1}{(s - \lambda)^2}Y_2(s) = \frac{a_1}{s - \lambda} + \frac{a_2}{(s - \lambda)^2} + \frac{a_3}{(s - \lambda)^3}.
$$

(c) Determine $y(t)$ by applying the inverse Laplace transform to $Y_1(s)$, $Y_2(s)$ and $Y_3(s)$.

**Solution:** The relevant inverse Laplace transforms are,

$$
\mathcal{L}^{-1}[1/(s - \lambda)] = e^{\lambda t},
$$

$$
\mathcal{L}^{-1}[1/(s - \lambda)^2] = te^{\lambda t},
$$

$$
\mathcal{L}^{-1}[1/(s - \lambda)^3] = \frac{1}{2!}t^2 e^{\lambda t}
$$

Therefore,

$$
\begin{align*}
y_1(t) &= a_1 e^{\lambda t} + a_2 t e^{\lambda t} + a_3 \frac{1}{2} t^2 e^{\lambda t}, \\
y_2(t) &= a_2 e^{\lambda t} + a_3 t e^{\lambda t}, \\
y_3(t) &= a_3 e^{\lambda t}
\end{align*}
$$

In matrix form, this is,

$$
y(t) = a_1 e^{\lambda t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 e^{\lambda t} \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + a_3 e^{\lambda t} \begin{bmatrix} \frac{t^2}{2} \\ t \\ 1 \end{bmatrix}.
$$
**Problem 5** For each of the following matrices $A$, compute the following,

(i) $\text{Trace}(A)$,
(ii) $\text{det}(A)$,
(iii) the characteristic polynomial $p_A(\lambda) = \text{det}(\lambda I - A)$,
(iv) the eigenvalues of $A$ (both real and complex), and
(v) for each eigenvalue $\lambda$ a basis for the space of $\lambda$-eigenvectors.

(a) The $2 \times 2$ matrix with real entries,

\[
A = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}.
\]

**Hint:** See Problem 3.

**Solution:** In Problem 3, we computed $\text{Trace}(A) = 0$, $\text{det}(A) = 1$, $p_A(\lambda) = \lambda^2 + 1$, and the eigenvalues are $\lambda_{\pm} = \pm i$. For the eigenvalue $\lambda_+ = i$, denote an eigenvector by,

\[
v_+ = \begin{bmatrix}
v_{+,1} \\
v_{+,2}
\end{bmatrix}.
\]

Then $-v_{+,2} = iv_{+,1}$, e.g., $v_{+,1} = 1$, $v_{+,2} = -i$. Therefore an eigenvector for $\lambda_+ = i$ is,

\[
v_+ = \begin{bmatrix}
1 \\
-i
\end{bmatrix}.
\]

Similarly, an eigenvector for $\lambda_- = -i$ is,

\[
v_- = \begin{bmatrix}
1 \\
i
\end{bmatrix}.
\]

(b) The $3 \times 3$ matrix with real entries,

\[
A = \begin{bmatrix}
3 & 1 & 1 \\
0 & 5 & 0 \\
0 & 0 & 3
\end{bmatrix}.
\]

**Solution:** Because this is an upper triangular matrix, clearly $\text{Trace}(A) = 3 + 5 + 3 = 11$, $\text{det}(A) = 3 \times 5 \times 3 = 45$, and $p_A(\lambda) = (\lambda - 3)(\lambda - 5)(\lambda - 3) = \lambda^3 - 11\lambda + 39\lambda - 45$. The eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 3$ (the eigenvalue 3 has multiplicity 2).

For the eigenvalue $\lambda_1 = 5$, the eigenvectors are the nonzero nullvectors of the matrix,

\[
A - 5I = \begin{bmatrix}
-2 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & -2
\end{bmatrix}.
\]

Either by using row operations to put this matrix in row echelon form, or by inspection, a basis for the nullspace is,

\[
v_1 = \begin{bmatrix}
1 \\
2 \\
0
\end{bmatrix}.
\]
For the eigenvalue $\lambda_2 = 3$, the eigenvectors are the nonzero nullvectors of the matrix,

$$A - 3I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

In this case, a basis for the nullspace is,

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$