**Problem 1** Let $r$ be a positive real number. Consider the 2nd order, linear differential equation,

$$y'' - \left( r + \frac{3}{t} \right) y' + \left( \frac{2r}{t} + \frac{3}{t^2} \right) y = 0,$$

where $y(t)$ is a function on $(0, \infty)$. One solution of this equation is $y_1(t) = te^{rt}$. Use Wronskian reduction of order to find a second solution $y_2(t)$.

**Solution** For the Wronskian $W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$, differentiating gives,

$$W' = -a(t)W = \left( r + \frac{3}{t} \right) W.$$

This is a separable equation whose solution is,

$$\ln(W) = rt + 3\ln(t) + C,$$

in other words,

$$W(t) = At^3e^{rt}.$$

Without loss of generality, take $A = 1$.

By definition $v = y_2(t)$ is a solution of the following 1st order ODE,

$$te^{rt}v' - (rt + 1)e^{rt}v = t^3e^{rt}.$$

Putting this in normal form,

$$v' + (-r - \frac{1}{t})v = t^2.$$

An integrating factor for this equation is,

$$u(t) = \exp \left[ \int_{t_0}^{t} (-r - \frac{1}{s}) ds \right]$$

$$= \exp \left[ -rt - \ln(t) + B \right]$$

$$= Ct^{-1}e^{-rt},$$

where $C$ is a constant. Set $C = 1$.

The integrating factor reduces the ODE to,

$$\left[ t^{-1}e^{-rt}v \right]' = te^{-rt}.$$

Integrating by parts, the antiderivative of $te^{-rt}$ is,

$$\int te^{-rt}dt = -\frac{1}{r^2} (rt + 1)e^{-rt} + E.$$

Hence,

$$t^{-1}e^{-rt}v = -\frac{1}{r^2} (rt + 1)e^{-rt} + E.$$

One solution is,

$$v(t) = -\frac{1}{r^2}t(rt + 1).$$
Of course any multiple of this solution also leads to a basic solution set. Therefore a basic solution set of the ODE,

\[ y'' - \left( r + \frac{3}{t} \right) y' + \left( \frac{2r}{t} + \frac{3}{t^2} \right) y = 0, \]

is the pair,

\[ y_1(t) = t e^r, \quad y_2(t) = t(\tau t + 1). \]

**Problem 2** An undamped harmonic oscillator satisfies the ODE,

\[ y'' + \omega^2 y = 0. \]

Let \( y(t) \) be a solution of this ODE for \( t < \tau \). At some time \( \tau > 0 \), the oscillator is given an *impulse* of size \( v > 0 \). In other words, if

\[ \begin{aligned} \lim_{\tau \to -\tau^-} y(t) &= y_0, \\ \lim_{\tau \to -\tau^+} y'(t) &= v_0 \end{aligned} \]

then for \( t > \tau \), \( y(t) \) is a solution of the IVP,

\[ \begin{aligned} y'' + \omega^2 y &= 0, \\ y(\tau) &= y_0, \\ y'(\tau) &= v_0 + v \end{aligned} \]

**(a)** Write \( y(t) \) in normal form \( A \cos(\omega t - \phi) \) for \( t < \tau \), and in normal form \( y(t) = B \cos(\omega t - \psi) \) for \( t > \tau \). Find an equation expressing \( B^2 \) in terms of \( A^2 \), \( v_0 \) and \( v \).

**Solution** For a function \( z(t) \) in the form \( C \cos(\omega t - \theta) \), the derivative is \( z'(t) = -\omega C \sin(\omega t - \theta) \).

In particular,

\[
(\omega z)^2 + (z')^2 = \omega^2 C^2 \cos^2(\omega t - \theta) + \omega^2 C^2 \sin^2(\omega t - \theta) = \omega^2 C^2.
\]

In particular,

\[
\omega^2 B^2 = (\omega y(\tau))^2 + (y'(\tau))^2 \\
= (\omega y_0)^2 + (v_0 + v)^2 = (\omega y_0)^2 + v_0^2 + 2v_0 v + v^2 \\
= \omega^2 A^2 + 2v_0 v + v^2.
\]

This gives the formula,

\[ B^2 = A^2 + 2 \frac{1}{\omega^2} v_0 v + \frac{1}{\omega^2} v^2. \]

**(b)** If the goal of the impulse is to maximize the amplitude \( B \), at what moment \( \tau \) in the cycle of the oscillator should the impulse be applied? If the goal is minimize the amplitude \( B \), at what moment \( \tau \) should the impulse be applied?

**Solution** Maximizing \( B \) is the same as maximizing \( B^2 \). In the equation above, \( A^2 \), \( \omega \) and \( v \) are the same for all values of \( \tau \). The only quantity that varies is \( v_0 \). To maximize \( B^2 \), the impulse should be applied when \( v_0 \) is as large as possible, at the moment when \( y_0 = 0 \) and \( y'(t) > 0 \). In other words, when

\[
\omega \tau - \phi = (2n - 1/2)\pi, \quad \tau = \frac{1}{\omega} (\phi + (2n - 1/2)\pi).
\]

Similarly, to minimize \( B \), the impulse should be applied when \( v_0 \) is as negative as possible, at the moment when \( y_0 = 0 \) and \( y'(t) < 0 \). In other words, when

\[
\omega \tau - \phi = (2n + 1/2)\pi, \quad \tau = \frac{1}{\omega} (\phi + (2n + 1/2)\pi).
\]

**Problem 3** Consider the following constant coefficient linear ODE,

\[ y''' + y = 0. \]

**(a)** Find the characteristic polynomial and find all real and complex roots.
Solution The characteristic polynomial is,\[ p(z) = z^3 + 1. \]

One evident root is \( z = -1 \). Factoring this out gives,
\[ z^3 + 1 = (z + 1)(z^2 - z + 1). \]

By the quadratic formula, the two roots of \( z^2 - z + 1 \) are the complex conjugates,
\[ \lambda_{\pm} = 1/2 \pm i\sqrt{3}/2. \]

(b) Find the general real-valued solution of the ODE.

Solution Associated to the root \(-1\) is the real-valued solution \( e^{-t} \). Associated to the complex conjugates \( \lambda_{\pm} \) are the two real solutions,
\[ e^{t/2} \cos(\sqrt{3}t/2), \quad e^{t/2} \sin(\sqrt{3}t/2). \]

Therefore the general real-valued solution is,
\[ y_g(t) = C_1 e^{-t} + C_2 e^{t/2} \cos(\sqrt{3}t/2) + C_3 e^{t/2} \sin(\sqrt{3}t/2). \]

(c) Find a particular solution of the driven ODE,
\[ y'' + y = \cos(\sqrt{3}t/2). \]

Solution A particular solution is the real part of the complex-valued solution of the driven complex ODE,
\[ \tilde{y}'' + \tilde{y} = e^{i\sqrt{3}t/2}. \]

Because \( i\sqrt{3}/2 \) is not a root of the characteristic polynomial, we guess the solution is of the form,
\[ \tilde{y} = Ae^{i\sqrt{3}t/2}. \]

Substituting this into the ODE gives,
\[ (i\sqrt{3}/2)^3 Ae^{i\sqrt{3}t/2} + Ae^{i\sqrt{3}t/2} = e^{i\sqrt{3}t/2}. \]

Simplifying gives,
\[ A(1 - 3\sqrt{3}i/8) = 1, \]

i.e.,
\[ \frac{1}{8} A(8 - 3\sqrt{3}i) = 1. \]

Multiplying both sides by the complex conjugate \( 8 + 3\sqrt{3}i \) gives,
\[ \frac{1}{8} A(64 - 27) = (8 + 3\sqrt{3}i), \]

i.e.
\[ A = \frac{8}{37} (8 + 3\sqrt{3}i). \]

So the real part of \( \tilde{y}(t) \) is,
\[ y_d(t) = \frac{8}{37} (8 \cos(\sqrt{3}t/2) - 3\sqrt{3} \sin(\sqrt{3}t/2)). \]

Problem 4 The linear ODE,
\[ y'' + (t - 3/t)y' - 2y = 0, \]

has a basic solution pair \( y_1(t) = e^{-t^2/2}, y_2(t) = t^2 - 2. \)

(a) Find the Wronskian \( W[y_1, y_2](t) \).
Solution Computing the derivatives,
\begin{align*}
y_1(t) &= e^{-t^2/2}, \quad y_2(t) = t^2 - 2, \\
y_1'(t) &= -te^{-t^2/2}, \quad y_2'(t) = 2t.
\end{align*}
So the Wronskian is,
\[2te^{-t^2/2} - (-t)(t^2 - 2)e^{-t^2/2} = t^3e^{-t^2/2}.
\]

(b) Use variation of parameters to find a particular solution of the driven ODE,
\[y'' + (t - 3/t)y' - 2y = t^4.
\]
Solution By variation of parameters, a particular solution of \(Ly = f(t)\) is,
\[y_d(t) = \int_{t_0}^{t} K(t, s)f(s)ds,
\]
where,
\[K(t, s) = (y_1(s)y_2(t) - y_1(t)y_2(s))/W[y_1, y_2](s).
\]
By (a), \(W(s) = s^3e^{-s^2/2}\). Therefore,
\[K(t, s) = (e^{-s^2/2}(t^2 - 2) - e^{-t^2/2}(s^2 - 2))/(s^3e^{-s^2/2}).
\]
Simplifying, this is,
\[K(t, s) = \frac{1}{s^3}(t^2 - 2) - e^{-t^2/2}\left(\frac{s^2 - 2}{s^3}\right) e^{s^2/2}.
\]
Multiplying by \(s^4\) yields,
\[K(t, s)s^4 = (t^2 - 2)s - e^{-t^2/2}(s^3 - 2s)e^{s^2/2}.
\]
The antiderivative of the first term is,
\[\int_{t_0}^{t} (t^2 - 2)sd(t) = \frac{1}{2}(t^2 - t_0^2)(t^2 - 2).
\]
To antidifferentiate the second term, substitute \(u = s^2/2, du = sds\) to get,
\[\int_{t_0^2/2}^{t^2/2} -e^{-t^2/2}(u - 2)e^u du.
\]
Integrating by parts, this is,
\[\int_{t_0^2/2}^{t^2/2} -e^{-t^2/2}(u - 2)e^u du =
- e^{-t^2/2}((u - 3)u|_{t_0^2/2}^{t^2/2} =
- e^{-t^2/2}\left(\frac{1}{2}(t^2 - 6)e^{t^2/2} - \frac{1}{2}(t_0^2 - 6)e^{t_0^2/2}\right) =
- \frac{1}{2}(t^2 - 6) + \frac{1}{2}(t_0^2 - 6)e^{t_0^2/2}e^{-t^2/2}.
\]
Putting the pieces together and plugging in \(t_0 = 0\) gives,
\[y_d(t) = \frac{1}{2}(t^4 - 3t^2 + 6) - 3e^{-t^2/2}.
\]
It is straightforward to check this is a solution.

Problem 5 Recall that \(PC_\mathbb{R}(0, 1]\) is the set of all piecewise continuous real-valued functions on the interval \((0, 1]\). The inner product on this set is,
\[\langle f, g \rangle = \int_0^1 f(t)g(t)dt.
\]
Define $f_0(t) = 1$. For each integer $n \geq 1$, define $f_n(t)$ to be the piecewise continuous function whose value on $[0, \frac{1}{2^n}]$ is $-1$, whose value on $[\frac{1}{2^n}, \frac{2}{2^n}]$ is $+1$, whose value on $[\frac{2}{2^n}, \frac{3}{2^n}]$ is $-1$, whose value on $[\frac{3}{2^n}, \frac{1}{2}]$ is $+1$, etc. In other words,

$$f_n(t) = \begin{cases} 
-1, & 2k-2 < t < 2k-2 \frac{2^n}{2^n} \text{ for } k = 1, \ldots, 2n-1, \\
1, & 2k-1 < t < 2k \frac{2^n}{2^n} \text{ for } k = 1, \ldots, 2n-1.
\end{cases}$$

(a) Compute the integrals $(f_m, f_n)$ and use this to prove that $(f_0, f_1, \ldots)$ is an orthonormal sequence. (Hint: If $n > m$, consider the integral of $f_n$ over one of the subintervals $(\frac{a}{2^n}, \frac{a+1}{2^n}]$. What fraction of the time is $f_n$ positive and what fraction of the time is it negative?)

**Solution** First of all, for every $n$, $(f_n(t))^2$ is the constant function $1$. Therefore $(f_n, f_n) = 1$. Suppose that $n > m$. Then the integral $(f_n, f_m)$ is the sum over all integers $a = 0, \ldots, 2^m - 1$ of the integral,

$$\int_{a/2^m}^{(a+1)/2^n} \pm f_n(t) \, dt.$$ 

Of course the interval $(\frac{a}{2^n}, \frac{a+1}{2^n}]$ is a union of $2^{m-n}$ intervals $(\frac{b}{2^n}, \frac{b+1}{2^n}]$. On half of these intervals, $f_n(t)$ has the constant value $-1$. On the other half, $f_n(t)$ has the constant value $+1$. Therefore the net integral of $f_n(t)$ over $(\frac{a}{2^n}, \frac{a+1}{2^n}]$ is $0$. Since this holds for each $a$,

$$(f_n, f_m) = 0.$$ 

Therefore the sequence $(f_0, f_1, \ldots)$ is an orthonormal sequence.

(b) Compute the generalized Fourier coefficient,

$$\langle t, f_n(t) \rangle = \int_0^1 t f_n(t) \, dt.$$ 

Prove it equals $\frac{1}{2^{n+1}}$. This gives the generalized Fourier series,

$$t = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f_n(t).$$

**Solution** Of course for $n = 0$, $\langle t, f_0(t) \rangle$ is just the integral of $t$, which is $\frac{1}{2}$. Suppose that $n > 0$. By definition,

$$\langle t, f_n(t) \rangle = \sum_{k=1}^{2^{n-1}} \left( \int_{(2k-2)/2^n}^{(2k-1)/2^n} t(-1) \, dt + \int_{(2k-1)/2^n}^{2k/2^n} t(1) \, dt \right).$$

Integrating, this is,

$$\sum_{k=1}^{2^{n-1}} \left( -\left(\frac{t^2}{2}\right)_{(2k-2)/2^n}^{(2k-1)/2^n} + \left(\frac{t^2}{2}\right)_{(2k-1)/2^n}^{2k/2^n} \right).$$

The term in parentheses simplifies to,

$$-\frac{1}{2} \left((2k-1)^2/2^{2n} - (2k-2)^2/2^{2n}\right) + \frac{1}{2} \left((2k)^2/2^{2n} - (2k-1)^2/2^{2n}\right) = \frac{1}{2^{2n+1}} \left(4k^2 - 2(2k-1)^2 + (2k-2)^2\right) = \frac{1}{2^{2n+1}} \left(4k^2 - 8k^2 + 8k - 2 + 4k^2 - 8k + 4\right) = \frac{1}{2^{2n+1}}.$$ 

Summing over all $k$ gives $2^{n-1} \times (1/2^{2n}) = 1/2^{n+1}$. Therefore the generalized Fourier coefficient is,

$$\langle t, f_n(t) \rangle = \frac{1}{2^{n+1}}.$$
This gives the generalized Fourier series,

\[ t = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f_n(t). \]

(c) Rewrite the series above as,

\[ t = \sum_{n=1}^{\infty} \frac{1 + f_n(t)}{2^n}. \]

What is the relationship of this equation to the binary expansion of the real number \( t \)?

Solution We can rewrite the equation because,

\[ \frac{1}{2} f_0 = \frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}}. \]

Now \( 1 + f_n(t) \) equals 0 iff the \( n \)th digit in the binary expansion of \( t \) equals 0. And \( 1 + f_n(t) \) equals 2 iff the \( n \)th digit in the binary expansion of \( t \) equals 1. Therefore \((1 + f_n(t))/2\) is precisely the \( n \)th digit in the binary expansion of \( t \). Therefore the formula above precisely says that \( t \) is equal to the series arising from the binary expansion of \( t \).