Limits in Iterated Integrals

3. Triple integrals in rectangular and cylindrical coordinates.

You do these the same way, basically. To supply limits for \( \iiint_D dz \, dy \, dx \) over
the region \( D \), we integrate first with respect to \( z \). Therefore we

1. Hold \( x \) and \( y \) fixed, and let \( z \) increase. This gives us a vertical line.

2. Integrate from the \( z \)-value where the vertical line enters the region \( D \) to the
\( z \)-value where it leaves \( D \).

3. Supply the remaining limits (in either \( xy \)-coordinates or polar coordinates)
so that you include all vertical lines which intersect \( D \). This means that you will
be integrating the remaining double integral over the region \( R \) in the \( xy \)-plane
which \( D \) projects onto.

For example, if \( D \) is the region lying between the two paraboloids

\[
\begin{align*}
  z &= x^2 + y^2 \\
  z &= 4 - x^2 - y^2,
\end{align*}
\]

we get by following steps 1 and 2,

\[
\iiint_D dz \, dy \, dx = \iint_R \int_{x^2+y^2}^{4-x^2-y^2} dz \, dA
\]

where \( R \) is the projection of \( D \) onto the \( xy \)-plane. To finish the job, we have to determine
what this projection is. From the picture, what we should determine is the \( xy \)-curve over
which the two surfaces intersect. We find this curve by eliminating \( z \) from the two equations,
getting

\[
\begin{align*}
  x^2 + y^2 &= 4 - x^2 - y^2, \\
  x^2 + y^2 &= 2.
\end{align*}
\]

Thus the \( xy \)-curve bounding \( R \) is the circle in the \( xy \)-plane with center at the origin and
radius \( \sqrt{2} \).

This makes it natural to finish the integral in polar coordinates. We get

\[
\iiint_D dz \, dy \, dx = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{x^2+y^2}^{4-x^2-y^2} dz \, r \, dr \, d\theta ;
\]

the limits on \( z \) will be replaced by \( r^2 \) and \( 4 - r^2 \) when the integration is carried out.