5. Triple Integrals

5A. Triple integrals in rectangular and cylindrical coordinates

5A-1 Evaluate: a) \( \int_{-1}^{1} \int_{0}^{1} (x + y + z) \, dx \, dy \, dz \) \quad b) \( \int_{0}^{2} \int_{0}^{\sqrt{y}} \int_{0}^{xy} 2xy^2 \, z \, dz \, dx \, dy \)

5A-2. Follow the three steps in the notes to supply limits for the triple integrals over the following regions of 3-space.

a) The rectangular prism having as its two bases the triangle in the \( yz \)-plane cut out by the two axes and the line \( y + z = 1 \), and the corresponding triangle in the plane \( x = 1 \) obtained by adding 1 to the \( x \)-coordinate of each point in the first triangle. Supply limits for three different orders of integration:

\[
\begin{align*}
(i) \quad & \iiint dz \, dy \, dx \\
(ii) \quad & \iiint dx \, dz \, dy \\
(iii) \quad & \iiint dy \, dx \, dz
\end{align*}
\]

b)* The tetrahedron having its four vertices at the origin, and the points on the three axes where respectively \( x = 1 \), \( y = 2 \), and \( z = 2 \). Use the order \( \iiint dz \, dy \, dx \).

c) The quarter of a solid circular cylinder of radius 1 and height 2 lying in the first octant, with its central axis the interval \( 0 \leq y \leq 2 \) on the \( y \)-axis, and base the quarter circle in the \( xz \)-plane with center at the origin, radius 1, and lying in the first quadrant. Integrate with respect to \( y \) first; use suitable cylindrical coordinates.

d) The region bounded below by the cone \( z^2 = x^2 + y^2 \), and above by the sphere of radius \( \sqrt{2} \) and center at the origin. Use cylindrical coordinates.

5A-3 Find the center of mass of the tetrahedron \( D \) in the first octant formed by the coordinate planes and the plane \( x + y + z = 1 \). Assume \( \delta = 1 \).

5A-4 A solid right circular cone of height \( h \) with 90° vertex angle has density at point \( P \) numerically equal to the distance from \( P \) to the central axis. Choosing the placement of the cone which will give the easiest integral, find

a) its mass \quad b) its center of mass

5A-5 An engine part is a solid \( S \) in the shape of an Egyptian-type pyramid having height 2 and a square base with diagonal \( D \) of length 2. Inside the engine it rotates about \( D \). Set up (but do not evaluate) an iterated integral giving its moment of inertia about \( D \). Assume \( \delta = 1 \). (Place \( S \) so the positive z axis is its central axis.)

5A-6 Using cylindrical coordinates, find the moment of inertia of a solid hemisphere \( D \) of radius \( a \) about the central axis perpendicular to the base of \( D \). Assume \( \delta = 1 \).

5A-7 The paraboloid \( z = x^2 + y^2 \) is shaped like a wine-glass, and the plane \( z = 2x \) slices off a finite piece \( D \) of the region above the paraboloid (i.e., inside the wine-glass). Find the moment of inertia of \( D \) about the \( z \)-axis, assuming \( \delta = 1 \).
5B. Triple Integrals in Spherical Coordinates

5B-1 Supply limits for iterated integrals in spherical coordinates \( \iiint d\rho \, d\phi \, d\theta \) for each of the following regions. (No integrand is specified; \( d\rho \, d\phi \, d\theta \) is given so as to determine the order of integration.)

a) The region of 5A-2d: bounded below by the cone \( z^2 = x^2 + y^2 \), and above by the sphere of radius \( \sqrt{2} \) and center at the origin.

b) The first octant.

c) That part of the sphere of radius 1 and center at \( z = 1 \) on the z-axis which lies above the plane \( z = 1 \).

5B-2 Find the center of mass of a hemisphere of radius \( a \), using spherical coordinates. Assume the density \( \delta = 1 \).

5B-3 A solid \( D \) is bounded below by a right circular cone whose generators have length \( a \) and make an angle \( \pi/6 \) with the central axis. It is bounded above by a portion of the sphere of radius \( a \) centered at the vertex of the cone. Find its moment of inertia about its central axis, assuming the density \( \delta \) at a point is numerically equal to the distance of the point from a plane through the vertex perpendicular to the central axis.

5B-4 Find the average distance of a point in a solid sphere of radius \( a \) from

- a) the center
- b) a fixed diameter
- c) a fixed plane through the center

5C. Gravitational Attraction

5C-1.* Find the gravitational attraction of the solid \( V \) bounded by a right circular cone of vertex angle \( 60^\circ \) and slant height \( a \), surmounted by the cap of a sphere of radius \( a \) centered at the vertex of the cone; take the density to be

(a) \( 1 \) \hspace{1cm} (b) the distance from the vertex. \hspace{1cm} Ans.: a) \( \pi Ga/4 \) \hspace{1cm} b) \( \pi Ga^2/8 \)

5C-2. Find the gravitational attraction of the region bounded above by the plane \( z = 2 \) and below by the cone \( z^2 = 4(x^2 + y^2) \), on a unit mass at the origin; take \( \delta = 1 \).

5C-3. Find the gravitational attraction of a solid sphere of radius 1 on a unit point mass \( Q \) on its surface, if the density of the sphere at \( P(x, y, z) \) is \( |PQ|^{-1/2} \).

5C-4. Find the gravitational attraction of the region which is bounded above by the sphere \( x^2 + y^2 + z^2 = 1 \) and below by the sphere \( x^2 + y^2 + z^2 = 2z \), on a unit mass at the origin. (Take \( \delta = 1.\))

5C-5.* Find the gravitational attraction of a solid hemisphere of radius \( a \) and density 1 on a unit point mass placed at its pole. \hspace{1cm} Ans: \( 2\pi Ga(1 - \sqrt{2}/3) \)
5C-6.* Let \( V \) be a uniform solid sphere of mass \( M \) and radius \( a \). Place a unit point mass a distance \( b \) from the center of \( V \). Show that the gravitational attraction of \( V \) on the point mass is

\[
\begin{align*}
\text{a) } & \frac{GM}{b^2}, \quad \text{if } b \geq a; \\
\text{b) } & \frac{GM'}{b^2}, \quad \text{if } b \leq a, \quad \text{where } M' = \frac{b^3}{a^3} M.
\end{align*}
\]

Part (a) is Newton’s theorem, described in the Remark. Part (b) says that the outer portion of the sphere—the spherical shell of inner radius \( b \) and outer radius \( a \)—exerts no force on the test mass: all of it comes from the inner sphere of radius \( b \), which has total mass \( \frac{b^3}{a^3} M \).

5C-7.* Use Problem 6b to show that if we dig a straight hole through the earth, it takes a point mass \( m \) a total of \( \pi \sqrt{R/g} \approx 42 \) minutes to fall from one end to the other, no matter what the length of the hole is.

(Write \( \mathbf{F} = ma \), letting \( x \) be the distance from the middle of the hole, and obtain an equation of simple harmonic motion for \( x(t) \). Here
\[
R = \text{earth’s radius}, \quad M = \text{earth’s mass}, \quad g = \frac{GM}{R^2}.
\]