DAVID JORDAN: Hello, and welcome back to recitation. So the problem that I want to work with you now is to compute some integrals, but we want to compute them in the presence of a density function.

So the region that we're considering is very simple. It's just the unit square. So we have the origin here, we have the line $x$ equals 1, we have the line $y$ equals 1, and we just want to compute in this region. And so we want to use this density function to find various sort of physical characteristics of this region.

So first, we want to find its mass, and so we are going to need to recall how you get mass from density. We want to find the center of mass. That is, where is the point on which we could balance this if we cut it out? If we tried to balance it on our fingers, where is the average mass concentrated? We want to find the moment of inertia about the origin, and we want to find the moment of inertia about the $x$-axis. So we're going to have to remember our formulas for moments of inertia.

So why don't you pause the video and work on this for a little bit. Check back with me and I'll show you how I solved it.

Hi. Welcome back. Why don't we start by finding the mass.

So the mass is the most straightforward of these, and I find it helpful to use the language of differentials. So what I want us to do is I want us to take a little square here, and this little square has area $dA$. OK? And what we want to do is we want to sum up the masses of all the little squares $dA$ here. So what we want to know is what is the little bit of mass $dM$ which corresponds to this little bit of area $dA$.

And more or less by definition, this is $\delta dA$. So $\delta$ is the ratio of area to mass. And so this little contribution of mass is just $\delta$ times the little contribution of area. OK? Now, once we write it this way, then our total mass for the entire square is just the integral over the region of all the little contributions of $dM$.

And so in particular, this is just the integral from $x$ equals 0 to 1, $y$ equals 0 to 1. We have $x^2y$-- that's our density-- and then we have $dy \, dx$. OK. And this is an integral which we can just compute. So why don't we compute this one all the way through and see what we get.

OK. So we have integral $x$ equals 0 to 1. So we have $x^2y$, and we need to integrate that in $y$. 
So we have \( x y^2 \) over 2. And then y ranges from 1 to 0, \( dx \). So this is the integral from
x equals 0 to 1 of \( x \) over 2, \( dx \). And this is just \( x^2 \) over 4 from 1 to 0. This is just \( 1/4 \). So
that tells us that the total mass of this unit square is \( 1/4 \).

OK. So now, we need to do similar-- we have a similar challenge for the other physical
quantities. We just need to figure out what is the appropriate differential quantity, and then we
just need to integrate that. For b, we need to compute the center of mass.

So remember that the center of mass involves finding the average x-coordinate and the
average y-coordinate. And I wanted to remind you what the formula is for the center of mass,
and remind you how I remind myself of it.

So in the formula for the center of mass, we need to take the average of \( x \) times \( dM \) divided by
the integral of \( dM \). So this is our formula for the center of mass. And I just wanted to say that
the way that I remember this is by thinking about seesaws.

So if you think about it-- and if we were not doing multiple variables but a single variable-- if I
had a seesaw, and I had some weights. So I had \( m_1 \) and \( m_2 \) and \( m_3 \) and \( m_4 \)-- I had
some weights-- and these were at positions \( x_1 \) and \( x_2 \) and \( x_3 \) and \( x_4 \). Well, the fact that
the scale would be balanced would be to say that this point \( x \) here, where the fulcrum is
located, is exactly the weighted average of these points. That's what's going to guarantee that
there's the same amount of torque pushing this way and this way. So if we were in one
variable and we just had some discrete weights, then we would want to take the average of all
of these positions, and we would want to weight it with the masses. So we would want to take
the sum of \( x_i m_i \) and divide by the sum of \( m_i \). This would be the average coordinates in this
kind of toy example.

And now if you look at the formula for the center of mass, it's really the same thing, isn't it?
Because integrals are just a continuous version of the sum. We have \( x \) as a function instead of
\( x_i \) as a list. And the \( m_i \)'s are just the little infinitesimal \( dM \)'s here. And then the bottom here is
just the total mass of the system, and so is this. OK, so that's how I think about this center of
mass formula. And it's actually pretty easy to compute.

So we have the integral from \( x \) equals 0 to 1, \( y \) equals 0 to 1. So now we have \( x \) times delta
times \( dx \) dy. So altogether we get \( x^2 \) dy dx.

So one of those \( x \)'s is because we're averaging \( x \) and the other one is from the density
function. So we have this whole integral. And then we divide by this integral of the mass, but we already computed this in part a, and we found it to be 1/4. OK. So this numerator here is fairly straightforward to compute. And if you do this you'll get-- let me double check-- I believe we got 1/6.

So you should get 1/6 when you compute this integral. So we have 1/6 over 1/4, and so cancelling off, this is 2/3. OK. So that was just the x center of mass.

But now I want to make an important point, which is that this density function is symmetric in x and y. It was just x times y. It wasn't something more complicated. And so the center of mass in the x-direction is just equal to the center of mass in the y-direction, so these are both equal to 2/3.

OK. So that depended on the fact that our density was symmetric, and also on the fact that our region was symmetric about switching x and y. So we could save ourselves some trouble here. OK, very good.

So now to do c, again we need to recall what is the infinitesimal moment of inertia. So let me draw this picture again. So here's our little dA here. And we want to know the infinitesimal moment of inertia around the origin. So we tie a string to this little piece of mass, and we start spinning it, and we want to know what is our moment of inertia corresponding to that little mass.

And I'll just remind you from lecture that the formula is r squared dM. So this is r squared times x*y dx dy. And so the r squared here is saying that as you get farther and farther out, your moment of inertia is getting larger and larger. And this makes sense in terms of the physical idea that you're moving a longer distance if you're farther out. So anyway. So this is our formula, r squared dM.

And so that tells us that I is just the integral of dI. And so this is the integral from x goes from 0 to 1, y goes from 0 to 1. And then we have x squared plus y squared-- that's just r squared-- times x*y dx dy. And so we can rewrite this as x cubed y plus x y cubed dx dy.

And this is a computation that we can do. Let me just check my notes real quick. So this is 1/4. I'll skip the computation, but this is just integrating some polynomials, so we can do that. All right.

And now finally, we want to compute the moment of inertia-- so remember, d asked us to
compute the moment of inertia around the x-axis. So instead of around the origin, it's around
the x-axis.

So the idea here is the same. So again, dl is a factor times dM. And again, it's the radius, but
now it's the radius about which we're spinning. So we're not anymore spinning around the
origin as we were doing before. Now we're spinning around-- sort of out of the board-- around
the x-axis here. But we still have the same formula, and now our radius is the height y.
Because we're not spinning around the origin anymore, we're spinning around this rod here.

And so if you think about it, that's the radius about which we're spinning is just the height y. So
this is just y squared delta. OK. And so that tells us that I-- the total inertia about the x-axis-- is
just the integral of dl. And so we get the integral from x equals 0 to 1, integral y equals 0 to 1.
And then we just have y squared x*y dy dx. And this, again, we could compute-- and let me
just check my notes-- and find that it's 1/8.

So in each of these problems, the most important thing to have been able to do is to argue,
what is this sort of infinitesimal contribution to the physical quantity that you want to compute?
And eventually, you want to express it in terms of the quantity dA, because dA is what we
actually can integrate. And so all the other physical quantities that we need to study are going
to be an integral of some infinitesimal element, and that infinitesimal element is going to be
some coefficient times dA.

So here, we had that this was-- oh, dear. This is a mistake. So this should have said y squared
dM, and that's y squared delta dA. So I wrote the delta implicitly. I wrote it twice.

So what we meant to say is dl is y squared delta dA. And so in all these examples, the
infinitesimal quantity that we're after is some straightforward coefficient times the infinitesimal
area. And so once we know that, then we can just do a straightforward integral.

OK, and I'll leave it at that.