Lagrange multipliers

1. In an open-top wooden drawer, the two sides and back cost $2/sq. ft., the bottom $1/sq. ft. and the front $4/sq. ft. Using Lagrange multipliers find the dimensions of the drawer with the largest capacity that can be made for $72.

Answer: The box shown has dimensions $x, y,$ and $z$.

The area of each side = $yz$; the area of the front (and back) = $xz$; the area of the bottom = $xy$. Thus, the cost of the wood is

$$C(x, y, z) = 2(2yz + xz) + xy + 4xz = 4yz + 6xz + xy = 72.$$  

This is our constraint. We are trying to maximize the volume

$$V = xyz.$$  

The Lagrange multiplier equations are then

$$\nabla V = \lambda \nabla C, \text{ and } C = 72$$

$$\iff (yz, xz, xy) = \lambda (6z + y, 4z + x, 4y + 6x), \quad 4yz + 6xz + xy = 72.$$  

We solve for the critical points by isolating $1/\lambda$.

$$\frac{1}{\lambda} = \frac{6}{y} + \frac{1}{z} = \frac{4}{x} + \frac{1}{z} = \frac{4}{x} + \frac{6}{y}$$

Comparing the third and fourth terms gives $\frac{1}{z} = \frac{6}{y} \iff y = 6z$.

Likewise the second and fourth terms give $x = 4z$.

Substituting this in the constraint gives $72z^2 = 72 \iff z = 1$. Thus,

$$z = 1, \quad x = 4, \quad y = 6.$$