Second derivative test

1. Find and classify all the critical points of

\[ w = (x^3 + 1)(y^3 + 1). \]

**Answer:** Taking the first partials and setting them to 0:

\[ w_x = 3x^2(y^3 + 1) = 0 \quad \text{and} \quad w_y = 3y^2(x^3 + 1) = 0. \]

The first equation implies \( x = 0 \) or \( y = -1 \). We use these one at a time in the second equation.

If \( x = 0 \) then \( w_y = 0 \) \( \Rightarrow y = 0 \) \( \Rightarrow (0,0) \) is a critical point.

If \( y = -1 \) then \( w_y = 0 \) \( \Rightarrow x^3 + 1 = 0 \) \( \Rightarrow x = -1 \) \( \Rightarrow (-1,-1) \) is a critical point.

The critical points are \((0,0)\) and \((-1,-1)\).

Taking second partials:

\[ w_{xx} = 6x(y^3 + 1), \quad w_{xy} = 9x^2y^2, \quad w_{yy} = 6y(x^3 + 1). \]

We analyze each critical point in turn.

At \((-1,-1)\): \( A = w_{xx}(-1,-1) = 0, \ B = w_{xy}(-1,-1) = 9, \ C = w_{yy}(-1,-1) = 0. \) Therefore \( AC - B^2 = -81 < 0 \), which implies the critical point is a saddle.

At \((0,0)\): \( A = w_{xx}(0,0) = 0, \ B = w_{xy}(0,0) = 0, \ C = w_{yy}(0,0) = 0. \) Therefore \( AC - B^2 = 0 \). The second derivative test fails.