Solutions to linear systems

1. Consider the system of equations
   \[
   \begin{align*}
   x + 2y + 3z &= 1 \\
   4x + 5y + 6z &= 2 \\
   7x + 8y + cz &= 3.
   \end{align*}
   \]

   a) Write the system in matrix form.

   b) For which values of \( c \) is there exactly one solution?

   c) For which values of \( c \) are there either 0 or infinitely many solutions?

   d) Take the corresponding homogeneous system
   \[
   \begin{align*}
   x + 2y + 3z &= 0 \\
   4x + 5y + 6z &= 0 \\
   7x + 8y + cz &= 0.
   \end{align*}
   \]

   For the value(s) of \( c \) found in part (c) give all the solutions.

   \textbf{Answer:} a) \( \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \).

   b) There is exactly one solution when the coefficient matrix has an inverse (i.e., is invertible). This happens when the determinant is not zero.

   \[
   \begin{vmatrix}
   1 & 2 & 3 \\
   4 & 5 & 6 \\
   7 & 8 & c
   \end{vmatrix} = 1(5c - 48) - 2(4c - 42) + 3(32 - 35) = -3c + 27 = 0 \iff c = 9.
   \]

   There is exactly one solution as long as \( c \neq 9 \).

   c) This is just the complement of part (b): there are zero or infinitely many solutions when \( c = 9 \).

   d) Setting \( c = 9 \) our coefficient matrix is \( A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \). Thinking of matrix multiplication as a series of dot products between rows of the left matrix and column(s) of the right one we see that in solving

   \[
   \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
   \]

   we are looking for vectors \( \langle x, y, z \rangle \) that are orthogonal to each of the rows of \( A \). Since \( \det(A) = 0 \), the rows are all in a plane and we can find orthogonal vectors by taking a cross product of (say) the first two rows.

   \[
   \langle 1, 2, 3 \rangle \times \langle 4, 5, 6 \rangle = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \langle -3, 6, -3 \rangle.
   \]
Since scaling will preserve orthogonality, all the solutions are scalar multiples, i.e., all the solutions are of the form \((x, y, z) = (-3a, 6a, -3a)\). We can make this a little nicer by removing the common factor of three,

\[(x, y, z) = (-a, 2a, -a) = a(-1, 2, -1).\]