Equations of planes

We have touched on equations of planes previously. Here we will fill in some of the details.

Planes in point-normal form

The basic data which determines a plane is a point \( P_0 \) in the plane and a vector \( \mathbf{N} \) orthogonal to the plane. We call \( \mathbf{N} \) a normal to the plane and we will sometimes say \( \mathbf{N} \) is normal to the plane, instead of orthogonal.

Now, suppose we want the equation of a plane and we have a point \( P_0 = (x_0, y_0, z_0) \) in the plane and a vector \( \overrightarrow{N} = \langle a, b, c \rangle \) normal to the plane.

Let \( P = (x, y, z) \) be an arbitrary point in the plane. Then the vector \( \overrightarrow{P_0P} \) is in the plane and therefore orthogonal to \( \mathbf{N} \). This means

\[
\mathbf{N} \cdot \overrightarrow{P_0P} = 0
\]

\[
\iff \langle a, b, c \rangle \cdot (x - x_0, y - y_0, z - z_0) = 0
\]

\[
\iff a(x - x_0) + b(y - y_0) + c(z - z_0) = 0
\]

We call this last equation the point-normal form for the plane.

Example 1: Find the plane through the point \((1,4,9)\) with normal \(\langle 2, 3, 4 \rangle\).

Answer: Point-normal form of the plane is \(2(x - 1) + 3(y - 4) + 4(z - 9) = 0\). We can also write this as \(2x + 3y + 4z = 50\).

Example 2: Find the plane containing the points \(P_1 = (1, 2, 3)\), \(P_2 = (0, 0, 3)\), \(P_3 = (2, 5, 5)\).

Answer: The goal is to find the basic data, i.e. a point in the plane and a normal to the plane. The point is easy, we already have three of them. To get the normal we note (see figure below) that \(\overrightarrow{P_1P_2}\) and \(\overrightarrow{P_1P_3}\) are vectors in the plane, so their cross product is orthogonal (normal) to the plane. That is,

\[
\mathbf{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{pmatrix} i & j & k \\ -1 & -2 & 0 \\ 1 & 3 & 2 \end{pmatrix} = -4i - j(-2) + k(-1) = \langle -4, 2, -1 \rangle.
\]
Using point-normal form (with point $P_1$) the equation of the plane is

$$-4(x - 1) + 2(y - 2) - (z - 3) = 0,$$

or equivalently

$$-4x + 2y - z = -3.$$ 

**Example 3:** Find the plane with normal $\mathbf{N} = \hat{k}$ containing the point (0,0,3)

**Eq. of plane:**

$$\langle 0,0,1 \rangle \cdot (x, y, z - 3) = 0 \iff z = 3.$$ 

**Example 4:** Find the plane with $x$, $y$ and $z$ intercepts $a$, $b$ and $c$.

**Answer:** We could find this using the method example 1. Instead, we’ll use a shortcut that works when all the intercepts are known. In this case, the intercepts are

$$(a,0,0), \quad (0,b,0), \quad (0,0,c)$$

and we simply write the plane as

$$x/a + y/b + z/c = 1.$$ 

You can easily check that each of the given points is on the plane.

For completeness we’ll do this using the more general method of example 1.

The 3 points give us 2 vectors in the plane, $\langle -a, b, 0 \rangle$ and $\langle -a, 0, c \rangle$.

$\Rightarrow \mathbf{N} = \langle -a, b, 0 \rangle \times \langle -a, 0, c \rangle = \langle bc, ac, ab \rangle$.

Point-normal form:

$$bc(x - a) + ac(y - 0) + ab(z - 0) = 0$$

$\iff bc x + ac y + ab z = abc \iff x/a + y/b + z/c = 1.$

**Lines in the plane**

While we’re at it, let’s look at two ways to write the equation of a line in the $xy$-plane.

**Slope-intercept form:** Given the slope $m$ and the $y$-intercept $b$ the equation of a line can be written $y = mx + b$.

**Point-normal form:**

We can also use point-normal form to find the equation of a line.

Given a point $(x_0, y_0)$ on the line and a vector $\langle a, b \rangle$ normal to the line the equation of the line can be written $a(x - x_0) + b(y - y_0) = 0$. 