Uses of dot product

1. Find the angle between \( \mathbf{i} + \mathbf{j} + 2 \mathbf{k} \) and \( 2 \mathbf{i} - \mathbf{j} + \mathbf{k} \).

   **Answer:** We call the angle \( \theta \) and use both ways of computing the dot product. Algebraically we have
   \[(\mathbf{i} + \mathbf{j} + 2 \mathbf{k}) \cdot (2 \mathbf{i} - \mathbf{j} + \mathbf{k}) = 2 - 1 + 2 = 3.\]

   Geometrically
   \[(\mathbf{i} + \mathbf{j} + 2 \mathbf{k}) \cdot (2 \mathbf{i} - \mathbf{j} + \mathbf{k}) = |\mathbf{i} + \mathbf{j} + 2 \mathbf{k}| \cdot |2 \mathbf{i} - \mathbf{j} + \mathbf{k}| \cos \theta = \sqrt{6} \sqrt{6} \cos \theta = 6.\]

   Combining these two we have
   \[6 \cos \theta = 3 \Rightarrow \cos \theta = \frac{3}{6} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.\]

2. a) Are \( \langle 1, 3 \rangle \) and \( \langle -2, 2 \rangle \) orthogonal?

   b) For what value of \( a \) are the vectors \( \langle 1, a \rangle \) and \( \langle 2, 3 \rangle \) at right angles?

   c) In the figure the vectors \( \mathbf{A} \) and \( \mathbf{B}_1 \) are orthogonal as are \( \mathbf{A} \) and \( \mathbf{B}_2 \). If all the vectors are the same length what are the coordinates of \( \mathbf{B}_1 \) and \( \mathbf{B}_2 \)?

   **Answer:** a) Vectors are orthogonal if their dot product is 0. So, taking the dot product
   \[\langle 1, 3 \rangle \cdot \langle -2, 2 \rangle = -2 + 6 = 4 \neq 0.\]

   Thus the vectors are not orthogonal.

   b) Setting the dot product to 0 and solving for \( a \) we get
   \[\langle 1, a \rangle \cdot \langle 2, 3 \rangle = 2 + 3a = 0 \Rightarrow a = -2/3.\]

   c) \( \mathbf{B}_1 \) is \( \mathbf{A} \) rotated \( 90^\circ \) clockwise. We will show that \( \mathbf{B}_1 = \langle a_2, -a_1 \rangle \). It is easy to check that
   \[|\langle a_2, -a_1 \rangle| = |\mathbf{A}| \text{ and } \langle a_2, -a_1 \rangle \cdot \mathbf{A} = 0.\]

   The figure above shows that putting the negative sign on the \( a_1 \) means \( \langle a_2, -a_1 \rangle \) is turned clockwise from \( \mathbf{A} \). Thus, \( \langle a_2, -a_1 \rangle = \mathbf{B}_1 \).

   \( \mathbf{B}_2 \) is \( \mathbf{A} \) rotated \( 90^\circ \) counterclockwise. Similarly to \( \mathbf{B}_1 \), we find \( \mathbf{B}_2 = \langle -a_2, a_1 \rangle \).
3. Using vectors and dot product show the diagonals of a parallelogram have equal lengths if and only if it’s a rectangle

**Answer:**

\[ \begin{align*}
D & \quad C \\
A & \quad B
\end{align*} \]

We will make use of two properties of the dot product

1. \( \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \).
2. \( \mathbf{v} \cdot \mathbf{w} = 0 \iff \mathbf{v} \perp \mathbf{w} \).

Referring to the figure, we will also need to use the fact that \( ABCD \) is a parallelogram. That is, \( \overrightarrow{AB} = \overrightarrow{DC} \).

We have \( \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} \) and \( \overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BC} - \overrightarrow{AB} \).

Taking dot products:

\[ |\overrightarrow{AC}|^2 = \overrightarrow{AC} \cdot \overrightarrow{AC} = (\overrightarrow{AB} + \overrightarrow{BC}) \cdot (\overrightarrow{AB} + \overrightarrow{BC}) = |\overrightarrow{AB}|^2 + 2\overrightarrow{AB} \cdot \overrightarrow{BC} + |\overrightarrow{BC}|^2. \]

and

\[ |\overrightarrow{BD}|^2 = \overrightarrow{BD} \cdot \overrightarrow{BD} + (\overrightarrow{BC} - \overrightarrow{AB}) \cdot (\overrightarrow{BC} - \overrightarrow{AB}) = |\overrightarrow{BC}|^2 - 2\overrightarrow{BC} \cdot \overrightarrow{AB} + |\overrightarrow{AB}|^2 \]

Comparing the two equations above we see

\[ |\overrightarrow{AC}|^2 = |\overrightarrow{BD}|^2 \iff 4\overrightarrow{AB} \cdot \overrightarrow{BC} = 0. \]

This shows the diagonals have the same length if and only if \( \overrightarrow{AB} \perp \overrightarrow{BC} \). That is, if and only if the sides of the parallelogram are orthogonal to each other. QED