1 Sum of independent random variables

**Lemma 1.** If $X$ and $Y$ are independent random variables, then

$$
\mathbb{P}(X + Y \leq z) = \mathbb{E}[F_X(z - Y)] = \mathbb{E}[F_Y(z - X)].
$$

**Proof.** We have

$$
\mathbb{P}(X + Y \leq z) = \mathbb{E}[1_{\{X+Y\leq z\}}]
= \int_{\mathbb{R}^2} 1_{\{x+y\leq z\}} d(\mathbb{P}_X \times \mathbb{P}_Y)(x,y)
= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} 1_{\{x+y\leq z\}} d\mathbb{P}_X(x) \right) d\mathbb{P}_Y(y)
= \int_{\mathbb{R}} F_X(z - y) d\mathbb{P}_Y(y)
= \mathbb{E}[F_X(z - Y)],
$$

where in the third inequality we used Fubini’s Theorem. \(\square\)

If $X$ and $Y$ are continuous, $X + Y$ is also continuous, and its density can be derived by differentiating the above expression, and using Exercise 7 of HW 5 to bring the differentiation inside the integral.

2 Gaussian, Gamma, and Exponential distributions

**Theorem 1.**

(a) If $N_1 \sim N(\mu_1, \sigma_1^2)$ and $N_2 \sim N(\mu_2, \sigma_2^2)$, then $N_1 + N_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

(b) If $G_1 \sim \text{Gamma}(k_1, \theta)$ and $G_2 \sim \text{Gamma}(k_2, \theta)$, then $G_1 + G_2 \sim \text{Gamma}(k_1 + k_2, \theta)$.

(c) If $N \sim N(0,1)$, then $N^2 \sim \text{Gamma}(1/2, 2)$.

(d) If $X, Y \sim N(0, 1)$, then $X^2 + Y^2 \sim \text{Exp}(2)$.

(e) If $X, Y \sim N(0, 1)$, then $\sqrt{X^2 + Y^2}$ and $\arcsin(Y/\sqrt{X^2 + Y^2})$ are independent. Furthermore, $\arcsin(Y/\sqrt{X^2 + Y^2})$ is uniform over $(-\pi/2, \pi/2)$.

**Proof.** (a) It follows from applying the convolution formula for continuous random variables, and doing lots of algebra. The whole thing is even in Wikipedia: 
(b) It also follows from applying the convolution formula, and doing some algebra. For the sake of simplicity, we prove it for the case $\theta = 1$.

\[
f_{G_1+G_2}(z) = \int_0^z f_{G_1}(x) f_{G_2}(z-x) \, dx \\
= \int_0^z x^{k_1-1} e^{-x} (z-x)^{k_2-1} e^{-(z-x)} \frac{dx}{\Gamma(k_1) \Gamma(k_2)} \\
= e^{-z} \int_0^z x^{k_1-1} (z-x)^{k_2-1} \frac{dx}{\Gamma(k_1) \Gamma(k_2)} \\
= e^{-z} z^{k_1+k_2-1} \Gamma(k_1) \int_0^1 t^{k_1-1} (1-t)^{k_2-1} \frac{dt}{\Gamma(k_1) \Gamma(k_2)} \\
= \frac{e^{-z} z^{k_1+k_2-1}}{\Gamma(k_1+k_2)}
\]

(c) We have

\[
\mathbb{P}(N^2 \leq z) = \mathbb{P}(|N| \leq \sqrt{z}) = 2 \int_0^{\sqrt{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt
\]

Then, differentiating with respect to $z$, we obtain

\[
f_{N^2}(z) = \frac{z^{\frac{1}{2}-1} e^{-\frac{z}{2}}}{\sqrt{2\pi}},
\]

which is the density of a $\text{Gamma}(1/2, 2)$.

(d) From (c), we know that $X^2$ and $Y^2$ are $\text{Gamma}(1/2, 2)$. Then, applying (b) we get that $X^2 + Y^2$ is $\text{Gamma}(1, 2)$, which is the same as $\text{Exp}(2)$.

(e) Note that $R = \sqrt{X^2 + Y^2}$ and $\Theta = \arcsin(Y/\sqrt{X^2 + Y^2})$ correspond to the radius and angle in polar coordinates. As a result, the probability of the event \(\{0 \leq \Theta \leq \theta_0\} \cap \{R \leq r_0\}\) can be computed using polar coordinates as follows:

\[
\mathbb{P}\left(\{0 \leq \Theta \leq \theta_0\} \cap \{R \leq r_0\}\right) = \int_{\{0 \leq \Theta \leq \theta_0\} \cap \{R \leq r_0\}} \frac{1}{2\pi} e^{\frac{-x^2+y^2}{2}} \, dxdy \\
= \int_0^{\theta_0} \int_0^{r_0} \frac{1}{2\pi} e^{\frac{-r^2}{2}} \, rdrd\theta \\
= \theta_0 \int_0^{r_0} \frac{1}{2\pi} e^{\frac{-r^2}{2}} \, dr \\
= \mathbb{P}(0 \leq \Theta \leq \theta_0) \mathbb{P}(R \leq r_0).
\]

Thus, they are independent, and $\Theta$ is uniform.

\[\square\]