Complements are independent too

**Problem 0.1.** Let \( \{A_i\}_{i \in T} \) be a (possibly infinite, possibly uncountable) set of independent events. Prove that \( \{A_i^c\}_{i \in T} \) is also independent.

Recall that independence means: for every finite \( I \subseteq T \),

\[
\mathbb{P}\left[ \bigcap_{i \in I} A_i \right] = \prod_{i \in I} \mathbb{P}[A_i]
\]

This can be done in other ways, e.g. by induction (though it’s a little bit of a pain) - the proof we’ll use involves the Inclusion-Exclusion formula.

**Proof.** What we want to prove is that

\[
\mathbb{P}\left[ \bigcap_{i} A_i^c \right] = \prod_{i} \mathbb{P}[A_i^c]
\]

given that the \( \{A_i\} \) are independent. We start by rewriting

\[
\prod_{i} \mathbb{P}[A_i^c] = \prod_{i} (1 - \mathbb{P}[A_i])
\]

\[
= 1 - \sum_{\text{all } i} \mathbb{P}[A_i] + \sum_{\text{all } (i,j)} \mathbb{P}[A_i] \mathbb{P}[A_j] - \sum_{\text{all } (i,j,k)} \mathbb{P}[A_i] \mathbb{P}[A_j] \mathbb{P}[A_k] \ldots
\]

(where “all \((i,j,k,\ldots)\)” refers only to unordered subsets of \([n]\)). By independence of \( \{A_i\} \) these products are just the probabilities of intersections, so (grouping the sum terms together) the above is

\[
1 - \left( \sum_{\text{all } i} \mathbb{P}[A_i] - \sum_{\text{all } (i,j)} \mathbb{P}[A_i \cap A_j] + \sum_{\text{all } (i,j,k)} \mathbb{P}[A_i \cap A_j \cap A_k] \ldots \right)
\]

But the thing inside the big parens is just the inclusion-exclusion formula! So we get

\[
= 1 - \mathbb{P}\left[ \bigcup_i A_i \right] = \mathbb{P}\left[ \left( \bigcup_i A_i \right)^c \right] = \mathbb{P}\left[ \bigcap_i A_i^c \right]
\]

and we are done. \(\square\)

**Remark:** This technique can also be used to show that changing any subset of the \( A_i \) to their complements also preserves independence.
Measuring probability of converging to an average density of $x$ heads

**Problem 0.2.** Consider the infinite-coin-toss model ($\Omega = \{0, 1\}^\infty$, and $\sigma$-algebra $\mathcal{F}$ developed in Lecture 2). Fix some $x \in [0, 1]$. Is the set of all sequences whose proportion of 1’s converges to $x$ measurable in $\mathcal{F}$?

**Proof.** First, we need to define our event. We call it

$$A_x := \left\{ \omega : \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i = x \right\}$$

To make this easier to work with, we note that $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i = x$ just means “for all $m \geq 1$, there exists some $N > 0$ (both $m, N$ are integers) such that

$$\text{for all } n \geq N, \quad \frac{1}{n} \sum_{i=1}^{n} \omega_i - x \leq \frac{1}{m}$$

We use this to define a collection of sets

$$S_{m,N} := \left\{ \omega : \text{for all } n \geq N, \quad \frac{1}{n} \sum_{i=1}^{n} \omega_i - x \leq \frac{1}{m} \right\}$$

Replacing “there exists” and “for all” with their equivalent set operations ($\cup$ and $\cap$ respectively) we get

$$A_x = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} S_{m,N}$$

So if we can show that $S_{m,N} \in \mathcal{F}$ for all $m, N$, we are done. To do so, let’s fix $m, N$ and define for all $k \geq 0$,

$$S_{m,N,k} := \left\{ \omega : \text{for } n \in \{N, N+1, \ldots, N+k\}, \quad \frac{1}{n} \sum_{i=1}^{n} \omega_i - x \leq \frac{1}{m} \right\}$$

Then we note two facts:

- $S_{m,N} = \bigcap_{k=0}^{\infty} S_{m,N,k}$;
- $S_{m,N,k} \in \mathcal{F}_{N+k} \subset \mathcal{F}_0$ (the algebra from which the $\sigma$-algebra $\mathcal{F}$ is built)

These facts together show that $S_{m,N} \in \sigma(\mathcal{F}_0) = \mathcal{F}$, and therefore $A_x \in \mathcal{F}$ as well. \qed
Of monkeys and typewriters: applying Borel-Cantelli

If you’ve ever heard the common statement that “a monkey at a typewriter will eventually write the entire works of Shakespeare (infinitely many times, no less)”, this is what it really means.

**Problem 0.3.** Suppose we have an infinite sequence of random coin flips - so \( \Omega = \{0,1\}^{\infty} \) - in which each coin flip is independent and has probability of producing 1 (“heads”) with probability \( p \in (0,1) \). Let \( b \in \{0,1\}^\ell \) be any finite pattern (so \( \ell \) is any positive integer). Prove that, almost surely, the pattern \( b \) occurs infinitely many times in the sequence.

To help prove this, we have the Borel-Cantelli lemma:

**Proposition 0.1 (Borel-Cantelli (part 2)).** Given a sequence \( A_n \) of events such that (i) \( \sum_n P[A_n] = \infty \) and (ii) the events \( \{A_n\} \) are independent, and defining \( A := \{A_n \ i.o.\} \) (note: see lecture 3 notes for the definition of this), then \( P[A] = 1 \).

**Proof.** The intuition is that we break up our outcome \( \omega \) into disjoint \( \ell \)-length blocks (running from bit \((n-1)\ell + 1\) to bit \(n\ell\) so the first block goes from 1 to \( \ell \)); letting \( b \) have \( j \) zeroes and \( k \) ones (\( j+k=\ell \)), and fixing a particular block \( \omega_{((n-1)\ell+1):(n\ell)} \), let \( A_n \) be the event that this block is actually equal to \( b \), i.e.

\[
A_n = \{ \omega : \omega_{((n-1)\ell+1):(n\ell)} = b \}
\]

Then, we have

\[
P[A_n] = (1-p)^j p^k > 0 \text{ (because } p \neq 0,1)\]

Therefore, \( \sum_n P[A_n] = \infty \); furthermore, the events \( \{A_n\} \) are independent because the blocks don’t overlap. So, almost surely, infinitely many of the \( A_n \) come true – and if this happens the sequence \( b \) occurs infinitely many times, as we wanted.

**Remark:** In reality, I have a hard time believing that a monkey in front of a typewriter will produce a sequence of independent letters, but for the sake of the metaphor we’ll pretend that it does.
Lebesgue measure on $\mathbb{R}$

See lecture notes (lecture 2).
EXTRA: Pairwise independence is not independence!

Not covered in recitation, and probably most people have already seen this, but something you should definitely know:

**Problem 0.4.** If a collection of events \( \{A_i\} \) are pairwise independent under a probability distribution (i.e. for any \( i \neq j \), \( P[A_i \cap A_j] = P[A_i]P[A_j] \)) are they necessarily independent as a collection?

No, they aren’t.

**Proof.** We’ll construct a simple counterexample in the two-fair-coins model (\( \Omega = \{0, 1\}^2, \mathcal{F} = 2^\Omega \), \( \mathbb{P} \) uniform). Let “\( \oplus \)” be the XOR operation, and define:

- \( A_1 := \{ \omega : \omega_1 = 1 \} \);
- \( A_2 := \{ \omega : \omega_2 = 1 \} \);
- \( A_{\oplus} := \{ \omega_1 \oplus \omega_2 = 1 \} \).

It is easy to check that each event has two elements, and so \( P[A_1] = P[A_2] = P[A_{\oplus}] = 1/2 \); it’s also easy to check that every pair of events is only satisfied by one elementary outcome (probability = 1/4), and so they are pairwise independent.

However, for them to be independent we would need \( P[A_1]P[A_2]P[A_{\oplus}] = P[A_1 \cap A_2 \cap A_{\oplus}] \) as well – but the left-hand side is 1/8 whereas the right-hand side is actually 0 because no event is in all three at once. \( \square \)