Boilerplate:
- No collaboration
- No internet, laptops, or cellphones
- Closed books, Closed notes.
- Cheat sheet allowed.
- Total: 100 pts

Exercise 1 (20 pts). In honor of Laos’ 65th independence day, some questions about independence.

1. Suppose $A$ is independent of $B$, $B$ is independent of $C$, and $C$ is independent of $A$. Is $A \cap C$ necessarily independent of $B$?
   Prove or give counterexample

2. What if, in the above part, we also have that $B \cap C$ is independent of $A$?
   Prove or give counterexample

3. Suppose $A_1 \supset A_2 \supset A_3 \supset \ldots$ and $A_n \rightarrow A$, and suppose that $B$ is independent of every $A_n$. Is $B$ necessarily independent of $A$?
   Prove or give counterexample

4. Suppose $A$ is independent of itself. What does this say about $\mathbb{P}(A)$?
   Don’t laugh - this actually happens and is crucial to prove certain very strong theorems, which we will go over later!

Bonus: No points, but a gold star for anyone who knows what country Laos gained independence from!

Exercise 2 15pts. Let $A_1, \ldots, A_n$ be events. Let $X(\omega)$ be the number of events that occurred when $\omega$ was the elementary outcome. Show

$$\sum_{1 \leq i < j \leq n} \mathbb{P}[A_i \cap A_j] = \mathbb{E} \left[ \frac{X(X-1)}{2} \right].$$
Use this to prove
\[ \mathbb{P}[\bigcup_{i=1}^{n} A_i] \geq \sum_{i=1}^{n} \mathbb{P}[A_i] - \sum_{1 \leq i < j \leq n} \mathbb{P}[A_i \cap A_j]. \]

Hint: rewrite this as \( \mathbb{E}[f(X)] \geq 0. \)

**Exercise 3** 15pts. Let \( X \) be a random variable taking values on non-negative integers \( \mathbb{Z}_+ = \{0, 1, \ldots\} \). It has the following amazing property: There is a constant \( c > 0 \) such that for any bounded function \( f : \mathbb{Z}_+ \rightarrow \mathbb{R} \) we have
\[ \mathbb{E}[f(X)] = c \mathbb{E}[X f(X - 1)]. \]

Note: \( c \) does not depend on \( f \).

Find distribution of \( X \).

**Exercise 4** 20pts. Let \( X, Y \) be two independent standard normal random variables \( \mathcal{N}(0, 1) \). Let \( Z = X^2 + Y^2 \). Recall (you don’t need to prove it) that \( Z \) has pdf \( f_Z(z) = \frac{1}{2} e^{-z/2} \), i.e. \( Z \sim \text{Exp}(1/2) \).

1. Show that if \( U \) is independent of \( Z \) and uniform on \([0, 2\pi)\) then \( \sqrt{Z} \sin mU \) is standard normal for any positive integer \( m \).

2. Show that \( T = \frac{2XY}{\sqrt{X^2+Y^2}} \) is standard normal. Hint: use polar coordinates

**Exercise 5** 30pts. Let \( X_1, X_2, \ldots \) be a sequence of i.i.d. Bernoulli random variables coin tosses, such that \( \mathbb{P}(X_1 = H) = p \in (0, 1) \). Let
\[ L_n = \max\{m \geq 0 : X_n = H, X_{n+1} = H, \ldots, X_{n+m-1} = H, X_{n+m} = T\} \]
be the length of the run of heads starting from the \( n \)-th coin toss. Prove that
\[ \limsup_{n \to \infty} \frac{L_n}{\log(n)} = \frac{1}{\log(1/p)} \quad \text{a.s.} \quad (1) \]

Steps:

1. Show that events \( \{L_n \geq r\}, \{L_{n+r} \geq r\}, \{L_{n+2r} \geq r\}, \ldots \) are jointly independent.

2. Show that for any random variables \( Z_n \)
\[ \mathbb{P}[Z_n > \beta\text{-i.o.}] = 0 \quad \Rightarrow \quad \limsup Z_n \leq \beta \text{ a.s.} \]
and
\[ \mathbb{P}[Z_n > \beta\text{-i.o.}] = 1 \quad \Rightarrow \quad \limsup Z_n \geq \beta \text{ a.s.} \]
