Readings:
(a) Notes from Lecture 6 and 7.
(b) [Cinlar] Sections I.4, I.5 and II.2
(c) [GS] Chapter 3

Exercise 1. Let \( N \) be a random variable that takes nonnegative integer values. Let \( X_1, X_2, \ldots \) be a sequence of i.i.d. discrete random variables that have finite expectation and are independent from \( N \). Use iterated expectations to show that the expected value of \( \sum_{i=1}^{N} X_i \) is \( \mathbb{E}[N] \mathbb{E}[X_1] \).

Exercise 2. Let \( X \) and \( Y \) be binomial with parameters \((m, p)\) and \((n, q)\), respectively.

(a) Show that if \( X \) is independent from \( Y \), \( m = n \), and \( p = q \) then \( X + Y \) is binomial. \textit{Hint:} Use the interpretation of the binomial, not algebra.

(b) Does the conclusion of part (a) remain valid if \( m \neq n \)? If \( X \) and \( Y \) are not independent? If \( p \neq q \)?

(c) Show that if \( X \) and \( Y \) are independent, then

\[
P(X + Y = k) = \sum_{i=-\infty}^{\infty} p_X(i) p_Y(k - i).
\]

(d) Use the result from part (c) to find the PMF of \( X + Y \) where \( X \) and \( Y \) are independent Poisson random variables with parameters \( \lambda \) and \( \mu \), respectively. \textit{Hint:} The “binomial theorem” states that

\[
(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.
\]

Exercise 3. A 4-sided die has its four faces labeled as \( a, b, c, d \). Each time the die is rolled, the result is \( a, b, c, \) or \( d \), with probabilities \( p_a, p_b, p_c, p_d \), respectively. Different rolls are statistically independent. The die is rolled \( n \) times. Let \( N_a \) and \( N_b \) be the number of rolls that resulted in \( a \) or \( b \), respectively. Find the covariance of \( N_a \) and \( N_b \).
Exercise 4. Suppose that $X$ and $Y$ are discrete random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. An elegant way of defining the conditional expectation of $Y$ given $X$ is as a random variable of the form $\phi(X)$ (where $\phi$ is a measurable function), such that

$$\mathbb{E}[\phi(X)g(X)] = \mathbb{E}[Yg(X)],$$

for all measurable functions $g$. In this problem, we will prove that this condition defines the conditional expectation uniquely; that is, if we also have

$$\mathbb{E}[\psi(X)g(X)] = \mathbb{E}[Yg(X)],$$

for every measurable function $g$, then $\phi(X)$ and $\psi(X)$ are almost surely equal, i.e., $\mathbb{P}(\phi(X) = \psi(X)) = 1$.

(a) Prove that the following sets are $\mathcal{F}$-measurable: $\{\phi(X) > \psi(X)\}$ and, for any integer $n$, $A_n := \{\phi(X) > \psi(X) + 1/n\}$.

(b) Assume the contradiction $\mathbb{P}(\phi(X) = \psi(X)) < 1$ and use $g(x) = 1_{A_n}$ for some appropriate $n$ to show that the conditional expectation is unique.

Exercise 5. A machine is refilled each morning with $n$ portions of vanilla and chocolate ice creams each (a total of $2n$ portions). Customers arrive sequentially, each getting one of the ice creams independently with probability $1/2$. Consider the first moment when a customer receives an “out of order” message. Let $X$ be the number of portions of the other type left at this moment, $0 \leq X \leq n$. Find the distribution of $X$.

Exercise 6. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. (So, $\mu$ is a measure, but not necessarily a probability measure.) Let $g : \Omega \to \mathbb{R}$ be a nonnegative measurable function. Let $\{B_i\}$ be a sequence of disjoint measurable sets. Prove that

$$\int_{\bigcup_i B_i} g \, d\mu = \sum_{i=1}^{\infty} \int_{B_i} g \, d\mu.$$  

(Be rigorous!)

Note: As an application, this exercise gives another rich source of probability measures. Namely, take $f$ – a nonnegative measurable function on the real line with $\int_{\mathbb{R}} f(x) \, dx = 1$ (integral w.r.t. Lebesgue measure), and define a set-function $\mathbb{P}(A) = \int_A f \, dx$. The exercise shows that $\mathbb{P}(\cdot)$ is a probability measure on $(\mathbb{R}, \mathcal{B})$. Function $f$ is called the probability density function (PDF) of $\mathbb{P}$. 

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Exercise 7. [Optional, not to be graded] Let $\mu$ and $\nu$ be two finite measures on $(\mathbb{R}, \mathcal{B})$. Show that if
\[
\int_{\mathbb{R}} f \, d\mu = \int_{\mathbb{R}} f \, d\nu
\]
for all bounded continuous functions $f$ then $\mu = \nu$. (Hint: write $\mathbb{1}_{(a,b)}(x)$ as an increasing limit of continuous functions.)

Note: This exercise shows that measure on Borel $\sigma$-algebra is uniquely characterized by its values on continuous functions. This is true on $\mathbb{R}, \mathbb{R}^n$ and any other topological space. Similar to how it is sufficient to know measures only on intervals $(-\infty, a)$ it is sufficient to consider only a handful of functions (such as all sines and cosines, or all exponents). This will be discussed later.