Readings:
Notes from Lecture 2 and 3.

Supplementary readings:
[GS], Sections 1.4-1.7.
[C], Chapter 1.3
[W], Chapter 1.

Exercise 1. Consider a probabilistic experiment involving infinitely many coin tosses, and let \( \Omega = \{0, 1\}^\infty \) (think of 0 and 1 corresponding to heads and tails, respectively). A typical element \( \omega \in \Omega \) is of the form \( \omega = (\omega_1, \omega_2, \ldots) \), with \( \omega_i \in \{0, 1\} \).

As in the notes for Lecture 2, we define \( F_n \) as the \( \sigma \)-field consisting of all sets whose occurrence or nonoccurrence can be determined by looking at the result of the first \( n \) coin flips. The \( \sigma \)-field \( \mathcal{F} \) for this model is defined as the smallest \( \sigma \)-field that contains all of the \( F_n \).

(a) Consider the event \( H \) consisting of all \( \omega \) with the following property.
There exists some time \( t \) at which the number of ones so far is greater than or equal to the number of zeros so far. Show that \( H \in \mathcal{F} \).

(b) (Harder) Consider the set \( A \) of all \( \omega \) for which the limit
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i
\]
exists. Show that \( A \in \mathcal{F} \).

Note: This is important because, once we have also chosen a probability measure, it allows us to make statements about the probability that this limit (the long-term fraction of heads) exists.

Hint: The event \( A_x \) “the limit defined above exists and is equal to \( x \)” belongs to \( \mathcal{F} \). However, this does not imply that \( \bigcup_x A_x \in \mathcal{F} \) (why?). You need to find some other way of describing the event \( A \) in terms of unions, complements, etc., of events in the \( F_n \). For example, use the fact that a sequence converges if and only if it is a “Cauchy sequence.”

Solution:
(a) Let \( S_n = \{ (\omega_1, \omega_2, \ldots) \mid \sum_{i=1}^{n} \omega_i \geq \lceil n/2 \rceil \} \), i.e., \( S_n \) is the set of sequences where there are at least as many ones, in the first \( n \) entries as there are zeroes. Then, \( H = \bigcup_{n=1}^{\infty} S_n \).

(b) Let \( a_n = \frac{1}{n} \sum_{i=1}^{n} \omega_i \).

According to Cauchy criterion, the sequence \( \{a_n\} \) converges if and only if for any positive integer \( r \), there exists some positive integer \( N \) such that for any \( n > m > N \),
\[
|a_n - a_m| < \frac{1}{r}.
\]

For a pair of positive integers \( n > m \), we define
\[
A_{1/r,n,m} = \left\{ \omega : \frac{1}{n} \sum_{i=1}^{n} \omega_i - \frac{1}{m} \sum_{i=1}^{m} \omega_i < \frac{1}{r} \right\} \in \mathcal{F}_n.
\]

Thus,
\[
A = \bigcap_{r=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{m=N}^{\infty} \bigcap_{n=m}^{\infty} A_{1/r,n,m} \in \mathcal{F}.
\]

**Exercise 2.** Suppose that the events \( A_n \) satisfy \( \mathbb{P}(A_n) \to 0 \) and \( \sum_{n=1}^{\infty} \mathbb{P}(A_n^c \cap A_{n+1}) < \infty \). Show that \( \mathbb{P}(A_n \text{ i.o.}) = 0 \). *Note:* \( A_n \text{ i.o.} \) stands for “\( A_n \) occurs infinitely often”, or “infinitely many of the \( A_n \) occur”, or just \( \limsup_n A_n \). *Hint:* Borel-Cantelli.

**Solution:** Define the set
\[
A = \limsup_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.
\]

We wish to show \( \mathbb{P}(A) = 0 \). Now, \( A \subseteq \bigcup_{m=n}^{\infty} A_m \) for all \( m \), and by monotonicity of the measure, \( \mathbb{P}(A) \leq \mathbb{P}(\bigcup_{m=n}^{\infty} A_m) \), for all \( n \). In addition,
\[
\bigcup_{m=n}^{\infty} A_m = A_n \cup (A_{n+1} \setminus A_n) \cup (A_{n+2} \setminus A_{n+1}) \cup \cdots
\]
\[
= A_n \cup (A_{n+1} \cap A_n^c) \cup (A_{n+2} \cap A_{n+1}^c) \cup \cdots.
\]
Therefore, by the union bound,

\[ P(A) \leq P \left( \bigcup_{m=n}^{\infty} A_m \right) \leq P(A_n) + \sum_{m=n}^{\infty} P(A_{m+1} \cap A_n^c). \]

This holds for all \( n \), and therefore it holds in the limit as \( n \) goes to infinity. But the limit of the final expression is zero, since \( P(A_n) \to 0 \), and since \( \sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}^c) < \infty \).

**Exercise 3.** Consider one of our standard probability spaces \((\Omega, \mathcal{F}, P)\), with \( \Omega = (0, 1] \), \( \mathcal{F} \) – Borel and \( P \) – the Lebesgue measure. To every element \( \omega \in \Omega \) we assign its infinite decimal representation. We disallow decimal representations that end with an infinite string of nines. Under this condition, every number has a unique decimal representation.

(a) Let \( A \) be the set of points in \((0, 1]\) whose decimal representation contains at least one digit equal to 9. Find \( P[A] \).

(b) Let \( B \) be the set of points that have infinitely many 9’s in the decimal representation. Find \( P[B] \). (Hint: Borel-Cantelli).

**Solution:** Part (a).

We will find the Lebesgue measure of \( A^c \), the set of points in \((0, 1]\) whose decimal representation contains no digit equal to 9. We can scale that set (by multiplying it with a real number) to obtain the set

\[ A_0 = \frac{1}{10} A^c, \]

which is the set of points in \((0, 1]\) whose decimal representation starts with a 0, and contains no digit equal to 9 afterwards. Since the set \( A_0 \) is just the same as \( A^c \) but scaled down by a factor of 10, we have that \( P(A_0) = \frac{1}{10} P(A^c) \). Furthermore, we can do translations of that set to obtain analogous sets starting with different digits. In particular, let us define

\[ A_k = \frac{k}{10} + A_0 \]
as the set of points in $(0, 1]$ whose decimal representation starts with a $k$, and has no digit equal to 9 afterwards. Note that these sets are all disjoint, and that we have

\[ A^c = \bigcup_{k=0}^{8} A_k. \]

Then, using the finite additivity property of measures, and the fact that the Lebesgue measure is invariant by translations, we obtain

\[
P(A^c) = \sum_{k=0}^{8} P(A_k) = \sum_{k=0}^{8} P(A_0) = \sum_{k=0}^{8} \frac{1}{10} P(A^c) = \frac{9}{10} P(A^c).
\]

This equality can only be true if $P(A^c) = 0$, and thus $P(A) = 1$.

Part (b). Let $B_i$ be the event that there is a 9 in the $i$-th position of the expansion. These events are independent with $P(B_i) = 1/10$, for all $i \geq 1$. Thus, we have

\[
\sum_{i=1}^{\infty} P(B_i) = \infty.
\]

Then, by Borel-Cantelli, we have

\[ P(B) = P(\{B_i \text{ i.o.}\}) = 1. \]

Exercise 4. Consider a probability space $(\Omega, \mathcal{F}, P)$, and let $A$ be an event (element of $\mathcal{F}$). Let $\mathcal{G}$ be collection of all events that are independent from $A$. Show that $\mathcal{G}$ need not be a $\sigma$-algebra.

Solution: $\mathcal{G}$ need not be a $\sigma$-algebra. For example, let $X, Y$ be i.i.d., with $P(X = 1) = P(X = 0) = 1/2$. Let $Z$ be the mod two sum of $X$ and $Y$, so that if $X = Y$, then $Z = 0$, and if $X \neq Y$, then $Z = 1$. Then pairwise, these three random variables are independent. Let $A$ be the event $\{Z = 1\}$. Now, the
events \( B_1 = \{ X = 1 \} \), \( B_2 = \{ Y = 1 \} \) are both independent of \( A \). However, \( B_1 \cap B_2 \) is not independent of \( A \).

**Exercise 5.** Let \( A_1, A_2, \ldots \) and \( B \) be events.

(a) Suppose that \( A_k \triangleleft A \), i.e. \( A_k \supset A_{k+1} \) and \( A = \bigcap_{k=1}^{\infty} A_k \). Assume \( B \) is independent of \( A_k \). Show that \( B \) is independent of \( A \).

(b) Suppose that \( A_1 \) is independent of \( B \) and also that \( A_2 \) is independent of \( B \). Is it true that \( A_1 \cap A_2 \) is independent of \( B \)? Prove or give a counterexample.

**Solution:**

(a) The sequence of events \( A_k \cap B \) is decreasing and converges to the event \( A \cap B \). [To see this, note that \( (\bigcap_{k\geq1} A_k) \cap B = \bigcap_{k\geq1} (A_k \cap B) \).] Using the continuity of probability measures in the first and last equalities below, and independence in the middle equality, we have

\[
P(A \cap B) = \lim_{k \to \infty} P(A_k \cap B) = \lim_{k \to \infty} P(A_k)P(B) = P(A)P(B).
\]

(b) Consider two independent and fair coin tosses and let \( A_i \) be the event that the \( i \)th toss results in heads. Let \( B \) be the event that both tosses give the same result. It is easily checked that \( P(A_i \cap B) = P(\{HH\}) = 1/4 = P(A_i)P(B) \), so that pairwise independence holds. On the other hand, \( P(B \mid A_1 \cap A_2) = 1 \neq P(B) \). Thus, \( A_1 \cap A_2 \) and \( B \) are not independent.

**Exercise 6.** Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space. Show that function

\[
d(A, B) \triangleq \mathbb{P}[A \Delta B]
\]

satisfies the triangle inequality (i.e. \( d(A, B) \leq d(A, C) + d(C, B) \) for any \( A, B, C \)).

*Fun fact:* Under this pseudo-metric any algebra is dense in the \( \sigma \)-algebra it generates. Thus, any event in a complicated \( \sigma \)-algebra (such as Borel) can be approximated arbitrarily well by events in a simple algebra (like finite unions of \([a, b])\).

**Solution:** The symmetric difference is \( A \Delta B = (A \setminus B) \cup (B \setminus A) \)

\[
A \Delta B = (A \setminus B) \cup (B \setminus A)
= (A \cap B^c) \cup (B \cap A^c)
= (A \cap B^c \cap C) \cup (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C) \cup (B \cap A^c \cap C^c)
\subset (C \setminus B) \cup (A \setminus C) \cup (C \setminus A) \cup (B \setminus C)
= (A \Delta C) \cup (C \Delta B).
\]
Hence, by the union bound,

$$\mathbb{P}(A \Delta B) \leq \mathbb{P}(A \Delta C) + \mathbb{P}(C \Delta B).$$

**Exercise 7.** [Optional, not to be graded] Let $\Omega_1 \subset \Omega$ and let $C$ be some collection of subsets of $\Omega$. Let

$$C_1 = C \cap \Omega_1 \triangleq \{ A \cap \Omega_1 : A \in C \}$$

and denote by $F_1$ ($F$) the minimal $\sigma$-algebra on $\Omega_1$ ($\Omega$) generated by $C_1$ ($C$). Also define

$$F_2 = F \cap \Omega_1 \triangleq \{ A \cap \Omega_1 : A \in F \}.$$ 

$F_2$ is called a trace of $F$ on $\Omega_1$. Show $F_1 = F_2$. (Hint: show that collection $G = \{ E \in F : E \cap \Omega_1 \in F_1 \}$ is a monotone class.)

**Solution:** For a collection $D$ and a space $\Omega$ let $\alpha_\Omega(D)$ denote the smallest algebra of sets in $\Omega$ containing $D$.

**Claim:** $\alpha_{\Omega_1}(C \cap \Omega_1) = \alpha_\Omega(C) \cap \Omega_1$.

By definition $C \subset \alpha_\Omega(C)$ and therefore, $C \cap \Omega_1 \subset \alpha_\Omega(C) \cap \Omega_1$. The empty set $\emptyset = \emptyset \cap \Omega_1 \in \alpha_\Omega(C) \cap \Omega_1$, as $\alpha_\Omega(C)$ is an algebra. Let $E \cap \Omega_1 \in \alpha_\Omega(C) \cap \Omega_1$, then $(E \cap \Omega_1)^c = \Omega_1 \setminus (E \cap \Omega_1) = E^c \cap \Omega_1 \in \alpha_\Omega(C) \cap \Omega_1$, as $E \in \alpha_\Omega(C)$ and $\alpha_\Omega(C)$ is an algebra. Let $E_1 \cap \Omega_1, E_2 \cap \Omega_1 \in \alpha_\Omega(C) \cap \Omega_1$, then $(E_1 \cap \Omega_1) \cap (E_2 \cap \Omega_1) = (E_1 \cap E_2) \cap \Omega_1 \in \alpha_\Omega(C) \cap \Omega_1$, as $E_1, E_2 \in \alpha_\Omega(C)$ and $\alpha_\Omega(C)$ is an algebra. Hence $\alpha_\Omega(C)$ is an algebra of sets in $\Omega_1$ containing $C \cap \Omega_1$, and by minimality of $\alpha_{\Omega_1}(C \cap \Omega_1), \alpha_{\Omega_1}(C \cap \Omega_1) \subset \alpha_\Omega(C) \cap \Omega_1$.

Consider the set

$$D_1 = \{ E \in 2^\Omega : E \cap \Omega_1 \in \alpha_{\Omega_1}(C \cap \Omega_1) \}.$$ 

The collection $C \subset D_1$, as $C \cap \Omega_1 \subset \alpha_{\Omega_1}(C \cap \Omega_1)$ by definition. The empty set $\emptyset = \emptyset \cap \Omega_1 = \emptyset \in \alpha_{\Omega_1}(C \cap \Omega_1)$, as $\alpha_{\Omega_1}$ is an algebra. Thus $\emptyset \in D_1$. Let $E \in D_1$, then $E^c \cap \Omega_1 = \Omega_1 \setminus (E \cap \Omega_1) = (E \cap \Omega_1)^c \in \alpha_{\Omega_1}(C \cap \Omega_1)$, as $E \cap \Omega_1 \in \alpha_{\Omega_1}(C \cap \Omega_1)$ and $\alpha_{\Omega_1}(C \cap \Omega_1)$ is an algebra. Thus $D_1$ is closed under complements. Let $E_1, E_2 \in D_1$, then $(E_1 \cap E_2) \cap \Omega_1 = (E_1 \cap \Omega_1) \cap (E_2 \cap \Omega_1) \in \alpha_{\Omega_1}(C \cap \Omega_1)$, as $E_1 \cap \Omega_1, E_2 \cap \Omega_1 \in \alpha_{\Omega_1}(C \cap \Omega_1)$ and $\alpha_{\Omega_1}(C \cap \Omega_1)$ is an algebra. Thus $D_1$ is closed under intersections and $D_1$ is an algebra of sets in $\Omega$ containing $C$. Therefore, by minimality $\alpha_\Omega(C) \subset D_1$. By definition of $D_1$, $\alpha_{\Omega_1}(C) \cap \Omega_1 \subset \alpha_{\Omega_1}(C \cap \Omega_1)$, which proves the claim.

**Claim:** For a collection of sets $D$ and a space $\Omega$, $\sigma_\Omega(D) = \sigma_{\Omega_1}(\alpha_{\Omega_1}(D))$. 


By definition $\mathcal{D} \subset \alpha_\Omega(\mathcal{D}) \subset \sigma_\Omega(\mathcal{D})$, and by monotonicity of the $\sigma_\Omega(\cdot)$ operator, see recitation 2, $\sigma_\Omega(\mathcal{D}) \subset \sigma_\Omega(\alpha_\Omega(\mathcal{D})) \subset \sigma_\Omega(\sigma_\Omega(\mathcal{D})) = \sigma_\Omega(\mathcal{D})$. Thus $\sigma_\Omega(\mathcal{D}) = \sigma_\Omega(\alpha_\Omega(\mathcal{D}))$.

Combining the results of the two claims $\sigma_\Omega_1(\alpha_\Omega(\mathcal{C}) \cap \Omega_1) = \sigma_\Omega_1(\alpha_\Omega(C \cap \Omega_1)) = \sigma_\Omega(C \cap \Omega_1)$ and $\sigma_\Omega(\alpha_\Omega(C)) = \sigma_\Omega(C)$. Therefore, it suffices to show that $\sigma_\Omega_1(\alpha_\Omega(\mathcal{C}) \cap \Omega_1) = \sigma_\Omega(\alpha_\Omega(C)) \cap \Omega_1$. By the monotone class theorem, as $\alpha_\Omega(C)$ is an algebra, this holds if and only if $\mu_\Omega(\alpha_\Omega(\mathcal{C}) \cap \Omega_1) = \mu_\Omega(\alpha_\Omega(C)) \cap \Omega_1$. Let $\mathcal{A} := \alpha_\Omega(C)$.

By definition $\mathcal{A} \subset \mu_\Omega(\mathcal{A})$ and therefore, $\mathcal{A} \cap \Omega_1 \subset \mu_\Omega(\mathcal{A}) \cap \Omega_1$. Let $\{E_n \cap \Omega_1\} \in \mu_\Omega(\mathcal{A}) \cap \Omega_1$, with $(E_n \cap \Omega_1) \subset (E_{n+1} \cap \Omega_1)$. The sequence $\{E_n\}$ may not be monotone however, $E_n = \cup_{k=1}^n E_n$ is monotonic and by the monotonicity of $\{E_n \cap \Omega_1\}$, $\cup_{k=1}^n E_k \cap \Omega_1 = E_n \cap \Omega_1$, i.e. $E_n \cap \Omega_1 = E_n' \cap \Omega_1$. Since $\mu_\Omega(\mathcal{A})$ is a monotone class $E_n' \nrightarrow E \in \mu_\Omega(\mathcal{A})$. Therefore, $E_n \cap \Omega_1 \nrightarrow E \in \mu_\Omega(\mathcal{A})' \cap \Omega_1$. Similarly, let $\{E_n \cap \Omega_1\} \in \mu_\Omega(\mathcal{A}) \cap \Omega_1$, with $(E_n \cap \Omega_1) \subset (E_{n+1} \cap \Omega_1)$, and, by the construction given for increasing sets, WLOG $E_n \supset E_{n+1}$. Since $\mu_\Omega(\mathcal{A})$ is a monotone class $E_n' \nRightarrow E \in \mu_\Omega(\mathcal{A})$. Therefore, $E_n \cap \Omega_1 \nRightarrow E \in \mu_\Omega(\mathcal{A}) \cap \Omega_1$, this follows since $\cup_{n=1}^\infty (E_n \cap \Omega_1) = (\cup_{n=1}^\infty E_n) \cap \Omega_1 = E \cap \Omega_1$. Hence $\mu_\Omega(\mathcal{A}) \cap \Omega_1$ is a monotone class of sets in $\Omega_1$ containing $\mathcal{A} \cap \Omega_1$ and by minimality $\mu_\Omega(\mathcal{A} \cap \Omega_1) \subset \mu_\Omega(\mathcal{A}) \cap \Omega_1$.

Consider the set

$$\mathcal{D}_2 = \{E \in 2^\Omega \mid E \cap \Omega_1 \in \mu_\Omega_1(\mathcal{A} \cap \Omega_1)\}.$$ 

The algebra $\mathcal{A} \subset \mathcal{D}_2$, as $\mathcal{A} \cap \Omega_1 \subset \alpha_\Omega(\mathcal{A} \cap \Omega_1)$ by definition. Let $\{E_n\}$ be an increasing sequence of sets in $\mathcal{D}_2$, then $\{E_n \cap \Omega_1\}$ is an increasing sequence of sets in $\mu_\Omega_1(\mathcal{A} \cap \Omega_1)$, and as $\mu_\Omega_1(\mathcal{A} \cap \Omega_1)$ is a monotone class, $(E_n \cap \Omega_1) \nrightarrow (E \cap \Omega_1) \in \mu_\Omega_1(\mathcal{A} \cap \Omega_1)$. Therefore, $E_n \nrightarrow E \in \mathcal{D}_2$. A similar argument holds for a decreasing sequence of sets. Hence $\mathcal{D}_2$ is a monotone class of sets in $\Omega$ containing $\mathcal{A}$. Therefore, by minimality $\mu_\Omega(\mathcal{A}) \subset \mathcal{D}_2$. By definition of $\mathcal{D}_2$, $\mu_\Omega(\mathcal{A}) \cap \Omega_1 \subset \mu_\Omega_1(\mathcal{A} \cap \Omega_1)$.

Hence $\mu_\Omega(\mathcal{A}) \cap \Omega_1 = \mu_\Omega_1(\mathcal{A} \cap \Omega_1)$, and thusly, $\sigma_\Omega(C) \cap \Omega_1 = \sigma_\Omega_1(C \cap \Omega_1)$ as desired.

**Exercise 8. [Optional, not to be graded]** Let $\Omega = [0,1]$ and let $\mathcal{F}_0$ be the collection of finite unions $\cup_{i=1}^N [a_i, b_i]$ for $a_i, b_i \in [0,1]$. For any $A \in \mathcal{F}_0$, let $\mathbb{P}[A] = 1$ if one of the $b_i = 1$, and $\mathbb{P}[A] = 0$ otherwise. In Lectures we showed that $\mathcal{F}_0$ is an algebra but not a $\sigma$-algebra.

(a) Show that $\mathbb{P}$ is a non-negative (finitely) additive set-function on $\mathcal{F}_0$. 

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(b) Show that $\mathbb{P}$ is not countably additive on $\mathcal{F}_0$.

**Solution:**

(a) For all $A \in \mathcal{F}_0$, $\mathbb{P}[A] \in \{0, 1\}$. Thus $\mathbb{P}$ is non-negative.

Let $A_1, A_2 \in \mathcal{F}_0$ be disjoint. Then $A_1 = \bigcup_{i=1}^{N_1} [a_i^{(1)}, b_i^{(1)}]$ and $A_2 = \bigcup_{j=1}^{N_2} [a_j^{(2)}, b_j^{(2)}]$, where WLOG the intervals are ordered and non are empty $a_1^{(m)} < b_1^{(m)} < \ldots < a_{N_m}^{(m)} < b_{N_m}^{(m)}$. As $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2 = \bigcup_{k=1}^{N_1+N_2} [a_k^{(3)}, b_k^{(3)}]$, where $a_k^{(3)} \in \{a_i^{(1)}, a_j^{(2)}\}$ and $b_k^{(3)} \in \{b_i^{(1)}, b_j^{(2)}\}$ are the results of interleaving the two collections of intervals and are again WLOG ordered. By construction $\mathbb{P}[A_1] = 1$ if and only if $b_1^{(1)} = 1$, $\mathbb{P}[A_2] = 1$ if and only if $b_2^{(2)} = 1$ and $\mathbb{P}[A_1 \cup A_2] = 1$ if and only if $b_{N_1+N_2}^{(3)} = 1$.

Moreover, $b_{N_1+N_2}^{(3)} = 1$ if and only if either $b_{N_1}^{(1)} = 1$ or $b_{N_2}^{(2)} = 1$. Suppose $b_{N_1+N_2}^{(3)} = 1$ and WLOG assume $b_{N_1}^{(1)} = 1$, then, as $A_1$ and $A_2$ are disjoint, $b_{N_2}^{(2)} \neq 1$.

$$\mathbb{P}(A_1 \cup A_2) = 1 = 1 + 0 = \mathbb{P}(A_1) + \mathbb{P}(A_2).$$

Suppose $b_{N_1+N_2}^{(3)} \neq 1$ then neither $b_{N_1}^{(1)} = 1$ nor $b_{N_2}^{(2)} = 1$.

$$\mathbb{P}(A_1 \cup A_2) = 0 = 0 + 0 = \mathbb{P}(A_1) + \mathbb{P}(A_2).$$

(b) Let $A_n = [0, 1 - \frac{1}{n})$. Then, for all $n$, $\mathbb{P}(A_n) = 0$. Moreover, $A_n \subset A_{n+1}$ and $\lim_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} A_n = [0, 1) \in \mathcal{F}_0$. Hence, by continuity of probability,

$$\lim_{n \to \infty} \mathbb{P}(A_n) = \lim_{n \to \infty} 0 = 0 \neq 1 = \mathbb{P}([0, 1)) = \mathbb{P} \lim_{n \to \infty} A_n,$$

and $\mathbb{P}$ is not countably additive.