Readings:
(a) Notes from Lecture 1.
(b) Handout on background material on sets and real analysis (Recitation 1).

Supplementary readings:
[C], Sections 1.1-1.4.
[GS], Sections 1.1-1.3.
[W], Sections 1.0-1.5, 1.9.

Exercise 1.
(a) Let $\mathbb{N}$ be the set of positive integers. A function $f : \mathbb{N} \to \{0, 1\}$ is said to be periodic if there exists some $N$ such that $f(n + N) = f(n)$, for all $n \in \mathbb{N}$. Show that the set of periodic functions is countable.

(b) Does the result from part (a) remain valid if we consider rational-valued periodic functions $f : \mathbb{N} \to \mathbb{Q}$?

Solution:
(a) For a given positive integer $N$, let $A_N$ denote the set of periodic functions with a period of $N$. For a given $N$, since the sequence, $f(1), \cdots, f(N)$, actually defines a periodic function in $A_N$, we have that each $A_N$ contains $2^N$ elements. For example, for $N = 2$, there are four functions in the set $A_2$:

$$f(1)f(2)f(3)f(4)\cdots = 0000\cdots; 1111\cdots; 0101\cdots; 1010\cdots.$$  

The set of periodic functions from $\mathbb{N}$ to $\{0, 1\}$, $A$, can be written as,

$$A = \bigcup_{N=1}^{\infty} A_N.$$  

Since the union of countably many finite sets is countable, we conclude that the set of periodic functions from $\mathbb{N}$ to $\{0, 1\}$ is countable.

(b) Still, for a given positive integer $N$, let $A_N$ denote the set of periodic functions with a period $N$. For a given $N$, since the sequence, $f(1), \cdots, f(N)$,
actually defines a periodic function in \( A_N \), we conclude that \( A_N \) has the same cardinality as \( Q^N \) (the Cartesian product of \( N \) sets of rational numbers). Since \( Q \) is countable, and the Cartesian product of finitely many countable sets is countable, we know that \( A_N \) is countable, for any given \( N \). Since the set of periodic functions from \( N \) to \( Q \) is the union of \( A_1, A_2, \ldots, \) it is countable, because the union of countably many countable sets is countable.

**Exercise 2.** Let \( \{x_n\} \) and \( \{y_n\} \) be real sequences that converge to \( x \) and \( y \), respectively. Provide a formal proof of the fact that \( x_n + y_n \) converges to \( x + y \).

**Solution:** Fix some \( \epsilon > 0 \). Let \( n_1 \) be such that \( |x_n - x| < \epsilon/2 \), for all \( n > n_1 \). Let \( n_2 \) be such that \( |y_n - y| < \epsilon/2 \), for all \( n > n_2 \). Let \( n_0 = \max\{n_1, n_2\} \). Then, for all \( n > n_0 \), we have

\[
| (x_n + y_n) - (x + y) | \leq |x_n - x| + |y_n - y| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]

which proves the desired result.

**Exercise 3.** We are given a function \( f : A \times B \to \mathbb{R} \), where \( A \) and \( B \) are nonempty sets.

(a) Assuming that the sets \( A \) and \( B \) are finite, show that

\[
\max_{x \in A} \min_{y \in B} f(x, y) \leq \min_{y \in B} \max_{x \in A} f(x, y).
\]

(b) For general nonempty sets (not necessarily finite), show that

\[
\sup_{x \in A} \inf_{y \in B} f(x, y) \leq \inf_{y \in B} \sup_{x \in A} f(x, y).
\]

**Solution:**

(a) The proof rests on the application of the following simple fact: if \( h(z) \leq g(z) \) for all \( z \) in some finite set \( Z \), then

\[
\min_{z \in Z} h(z) \leq \min_{z \in Z} g(z) \quad (1)
\]

\[
\max_{z \in Z} h(z) \leq \max_{z \in Z} g(z). \quad (2)
\]

Observe that for all \( x, y \),

\[
f(x, y) \leq \max_{x \in A} f(x, y),
\]

2
and Eq. (1) implies that for each $x$,
\[
\min_{y \in B} f(x, y) \leq \min_{y \in B} \max_{x \in A} f(x, y).
\]

Now applying Eq. (2), let’s take a maximum of both sides with respect to $x \in A$. Since the right-hand side is a number, it remains unchanged:
\[
\max_{x \in A} \min_{y \in B} f(x, y) \leq \min_{y \in B} \max_{x \in A} f(x, y),
\]
which is what we needed to show.

(b) Along the same lines, we have the fact that if $h(z) \leq g(z)$ for all $z \in Z$,
\[
\begin{align*}
\inf_{z \in Z} h(z) &\leq \inf_{z \in Z} g(z) \\
\sup_{z \in Z} h(z) &\leq \sup_{z \in Z} g(z).
\end{align*}
\]
These follow immediately from the definitions of \( \sup \) and \( \inf \).

As before, we begin with
\[
f(x, y) \leq \sup_{x \in A} f(x, y),
\]
for all $x, y$. By Eq. (3), for each $x$,
\[
\inf_{y \in B} f(x, y) \leq \inf_{y \in B} \sup_{x \in A} f(x, y),
\]
and using Eq. (4),
\[
\sup_{x \in A} \inf_{y \in B} f(x, y) \leq \inf_{y \in B} \sup_{x \in A} f(x, y).
\]

Exercise 4. A probabilistic experiment involves an infinite sequence of trials. For $k = 1, 2, \ldots$, let $A_k$ be the event that the $k$th trial was a success. Write down a set-theoretic expression that describes the following event:

$B$: For every $k$ there exists an $\ell$ such that trials $k\ell$ and $k\ell^2$ were both successes.

Note: A “set theoretic expression” is an expression like $\bigcup_{k>5} \bigcap_{\ell<k} A_{k+\ell}.$

Solution: $B = \bigcap_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} (A_{k\ell} \cap A_{k\ell^2}).$
Exercise 5. Let \( f_n, f, g : [0, 1] \to [0, 1] \) and \( a, b, c, d \in [0, 1] \). Derive the following set theoretic expressions:

(a) Show that
\[
\{ x \in [0, 1] \mid \sup_n f_n(x) \leq a \} = \{ x \in [0, 1] \mid f_n(x) \leq a \},
\]
and use this to express \( \{ x \in [0, 1] \mid \sup_n f_n(x) < a \} \) as a countable combination (countable unions, countable intersections and complements) of sets of the form \( \{ x \in [0, 1] \mid f_n(x) \leq b \} \).

(b) Express \( \{ x \in [0, 1] \mid f(x) > g(x) \} \) as a countable combination of sets of the form \( \{ x \in [0, 1] \mid f(x) > c \} \) and \( \{ x \in [0, 1] \mid g(x) < d \} \).

(c) Express \( \{ x \in [0, 1] \mid \limsup_n f_n(x) \leq c \} \) as a countable combination of sets of the form \( \{ x \in [0, 1] \mid f_n(x) \leq c \} \).

(d) Express \( \{ x \in [0, 1] \mid \lim_n f_n(x) \text{ exists} \} \) as a countable combination of sets of the form \( \{ x \in [0, 1] \mid f_n(x) < c \}, \{ x \in [0, 1] \mid f_n(x) > c \} \), etc. (Hint: think of \( \{ x \in [0, 1] \mid \limsup_n f_n(x) > \liminf_n f_n(x) \} \)).

Solution: First observe the following set relations
\[
[0, c) = \bigcup_{n=1}^{\infty} [0, c - \frac{1}{n}] \quad [0, c] = \bigcup_{n=1}^{\infty} [0, c + \frac{1}{n}]
\]
\[
(c, 1] = \bigcup_{n=1}^{\infty} [c + \frac{1}{n}, 1] \quad [c, 1] = \bigcup_{n=1}^{\infty} (c - \frac{1}{n}, 1].
\]

All conversions between strict and non-strict inequalities following from these relations and properties of the inverse image, i.e. homomorphism of arbitrary set operations. We will use the shorthand notation
\[
\{ f < a \} := \{ x \in [0, 1] \mid f(x) < a \}.
\]

(a) Let \( x \in \bigcap_n \{ f_n \leq a \} \). Then, \( f_n(x) \leq a \) for all \( n \) \( \implies \) \( \sup_n f_n(x) \leq a \), by definition of \( \sup \) as \( a \) is an upper bound for \( \{ f_n(x) \} \). Therefore, as \( x \) was arbitrary,
\[
\{ f_n \leq a \} \subseteq \{ \sup_n f_n \leq a \}.
\]
Let \( x \in \{ \sup_n f_n \leq a \} \). Then \( \sup_n f_n(x) \leq a \) and for all \( n f_n(x) \leq \sup_n(x) \leq a \). Therefore, as \( x \) was arbitrary,

\[
\{ \sup_n f_n \leq a \} \subset \{ f_n \leq a \}.
\]

Hence \( \{ \sup_n f_n \leq a \} = \bigcap_n \{ f_n \leq a \} \). By De Morgan’s this relation also implies

\[
\{ \sup_n f_n > a \} = \bigcup_{n=1}^\infty \{ f_n \geq a \}.
\]

Similar results hold for \( \inf \).

Let \( f = \sup_n f_n \). Using the above comment

\[
\{ \sup_n f_n < a \} = \{ f < a \}
\]
\[
= f^{-1}((0, a))
\]
\[
= \bigcup_{k=1}^\infty f^{-1} \left[ 0, a - \frac{1}{k} \right]
\]
\[
= \bigcup_{k=1}^\infty \{ \sup_n f_n \leq a - \frac{1}{k} \}
\]
\[
= \bigcup_{k=1}^\infty \{ f_n \leq a - \frac{1}{k} \}.
\]

(b) Using countability and density of the rationals

\[
\{ f > g \} = \bigcup_{q \in \mathbb{Q}} \{ f > q \} \cap \{ q > g \}
\]
\[
= \bigcup_{q \in \mathbb{Q}} \{ f > q \} \cap \{ q \geq g \}
\]
\[
= \bigcup_{q \in \mathbb{Q}} \{ f \geq q \} \cap \{ q > g \}.
\]
(c) 
\[
\{ \limsup_{n \to \infty} f_n \leq c \} = \{ \inf_{n \geq 1} \sup_{k \geq n} f_k \leq c \}
\]
\[
= \{ \inf_{m=1}^{\infty} \sup_{n \geq 1, k \geq n} f_k < c + \frac{1}{m} \}
\]
\[
= \bigcup_{m=1}^{\infty} \{ \sup_{k \geq n} f_k < c + \frac{1}{m} \}
\]
\[
= \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{k \geq n} \{ f_k \leq c + \frac{1}{m} - \frac{1}{n} \}
\]

(d) 
\[
\{ \lim_{n \to \infty} f_n \text{ exists} \} = \{ \liminf_{n \to \infty} f_n = \limsup_{n \to \infty} f_n \}
\]
\[
= \{ \liminf_{n \to \infty} f_n < \limsup_{n \to \infty} f_n \}^{c} \quad \text{(part b)}
\]
\[
= \left( \bigcup_{q \in Q} \{ \liminf_{n \to \infty} f_n < q \} \cap \{ q < \limsup_{n \to \infty} f_n \} \right) \quad \text{(part b)}
\]
\[
= \{ \liminf_{n \to \infty} f_n \geq q \} \cup \{ \limsup_{n \to \infty} f_n \leq q \}.
\]

The sets \( \{ \liminf_{n \to \infty} f_n \geq q \} \) and \( \{ \limsup_{n \to \infty} f_n \leq q \} \) can be expressed as countable combinations using part (c) and the fact that 
\[
- \limsup_{n \to \infty} f_n(x) = - \inf_{n \geq 1, k \geq n} \sup f_k(x)
\]
\[
= \sup_{n \geq 1, k \geq n} ( -f_k(x) )
\]
\[
= \lim_{n \to \infty} (-f_n(x)),
\]
i.e. \( \{ \liminf_{n \to \infty} f_n \geq q \} = \{ \limsup_{n \to \infty} (-f_n) \leq -q \} \). More specifically, 
\[
\bigcap_{q \in Q} \left[ \left( \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{\ell=1}^{\infty} \bigcup_{k=n}^{\infty} \{ f_k \geq c - \frac{1}{m} + \frac{1}{\ell} \} \right) \cup \left( \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{\ell=1}^{\infty} \bigcup_{k=n}^{\infty} \{ f_k \leq c + \frac{1}{m} - \frac{1}{\ell} \} \right) \right].
\]
Using one of the later two expressions of part (b), we can drop one of the outer intersections

$$\bigcap_{q \in \mathbb{Q}} \left[ \left( \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{ f_k \geq c + \frac{1}{m} \} \right) \cup \left( \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{ f_k \leq c - \frac{1}{m} \} \right) \right]$$

or

$$\bigcap_{q \in \mathbb{Q}} \left[ \left( \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{ f_k \geq c - \frac{1}{m} + \frac{1}{T} \} \right) \cup \left( \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \bigcup_{n=1}^{\infty} \{ f_k \leq c - \frac{1}{m} \} \right) \right].$$

Exercise 6. Let $\Omega = \mathbb{N}$ (the positive integers), and let $\mathcal{F}_0$ be the collection of subsets of $\Omega$ that either have finite cardinality or their complement has finite cardinality. For any $A \in \mathcal{F}_0$, let $P(A) = 0$ if $A$ is finite, and $P(A) = 1$ if $A^c$ is finite.

(a) Show that $\mathcal{F}_0$ is a field but not a $\sigma$-field.

(b) Show that $P$ is finitely additive on $\mathcal{F}_0$; that is, if $A, B \in \mathcal{F}_0$, and $A, B$ are disjoint, then $P(A \cup B) = P(A) + P(B)$.

(c) Show that $P$ is not countably additive on $\mathcal{F}_0$; that is, construct a sequence of disjoint sets $A_i \in \mathcal{F}_0$ such that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_0$ and $P \left( \bigcup_{i=1}^{\infty} A_i \right) \neq \sum_{i=1}^{\infty} P(A_i)$.

(d) Construct a decreasing sequence of sets $A_i \in \mathcal{F}_0$ such that $\bigcap_{i=1}^{\infty} A_i = \emptyset$ for which $\lim_{i \to \infty} P(A_i) = 0$.

Solution:

(a) The empty set has zero cardinality, and therefore belongs to $\mathcal{F}_0$. Furthermore, if $A \in \mathcal{F}_0$, then either $A$ or $A^c$ has finite cardinality. It follows that either $A^c$ or $(A^c)^c$ has finite cardinality, so that $A^c \in \mathcal{F}_0$.

Suppose that $A, B \in \mathcal{F}_0$. If both $A$ and $B$ are finite, then $A \cup B$ is also finite and belongs to $\mathcal{F}_0$. Suppose now that at least one of $A$ or $B$ is infinite. We have $A \cup B = (A^c \cap B^c)^c$. Since $A^c \cap B^c$ is finite, it follows that $A \cup B \in \mathcal{F}_0$. This shows that $\mathcal{F}_0$ is a field.

To see that $\mathcal{F}_0$ is not a $\sigma$-field, note that $\{2n\} \in \mathcal{F}_0$ for every $n \in \mathbb{N}$, but the set $\bigcup_{n=0}^{\infty} \{2n\}$, the set of even natural numbers, is not in $\mathcal{F}_0$.

(b) Let $A, B \in \mathcal{F}_0$ be disjoint. If both $A$ and $B$ are finite, then $P(A \cup B) = 0 = P(A) + P(B)$. Suppose that either $A$ or $B$ (or both) is infinite. Since $A$ and $B$ are disjoint, we have $A \subset B^c$ and $B \subset A^c$. It follows that $A$ and $B$ cannot both be infinite. Therefore, $P(A \cup B) = 1 = P(A) + P(B)$, and $P$ is finitely additive.
(c) Note that \( \{n\} \in \mathcal{F}_0 \) and \( \bigcup_{n \geq 1} \{n\} = \Omega \). However, \( \mathbb{P}(\{n\}) = 0 \) while \( \mathbb{P}(\Omega) = 1 \), hence \( \mathbb{P} \) is not countably additive.

(d) Let \( A_n = \{n, n + 1, \ldots\} \). Then \((A_n)_{n \geq 1}\) forms a decreasing sequence of sets with \( \bigcap_n A_n = \emptyset \). But \( \mathbb{P}(A_n) = 1 \) for all \( n \), hence \( \lim_{n \to \infty} \mathbb{P}(A_n) = 1 \).