Exercise 1. A particle performs a random walk on the vertex set of a finite connected undirected graph $G$, which for simplicity we assume to have neither self-loops nor multiple edges. At each stage it moves to a neighbor of its current position, each such neighbor being chosen with equal probability. If $G$ has $\eta$ edges, show that the stationary distribution is given by $\pi_v = d_v/(2\eta)$, where $d_v$ is the degree of each vertex $v$.

Exercise 2. A particle performs a random walk on a bow tie $ABCDE$ drawn on Figure 1, where $C$ is the knot. From any vertex, its next step is equally likely to be to any neighboring vertex. Initially it is at $A$. Find the expected value of:

(a) The time of first return to $A$.

(b) The number of visits to $D$ before returning to $A$.

(c) The number of visits to $C$ before returning to $A$.

(d) The time of first return to $A$, given that there were no visits to $E$ before the return to $A$.

(e) The number of visits to $D$ before returning to $A$, given that there were no visits to $E$ before the return to $A$.

![Figure 1: A bow tie graph.](image)

Exercise 3. Let $(\Omega, \mathcal{F}) = (\mathbb{R}^\infty, \mathcal{B}^\infty)$, $X_k(\omega) = \omega_k$, $k \in \mathbb{N}$, be the canonical coordinate functions, and $\mathcal{F}_k = \sigma(X_1, \ldots, X_k)$, $k \in \mathbb{N}$, be the natural filtration of this space. We say that $\tau$ is a stopping time of the filtration $\{\mathcal{F}_k\}$ if

(a) $\tau$ is a positive integer
(b) for every \( k \geq 1 \) we have \( \{ \tau \leq k \} \in \mathcal{F}_k \)

Let \( \tau : \Omega \to \mathbb{N} \) be \((\mathcal{F}, \mathcal{B})\) measurable. Show that \( \tau \) is a stopping of \( \{ \mathcal{F}_k \} \) if and only if for every \( \omega, \omega' \in \Omega \) and for every \( n \geq 1 \)

\[
\tau(\omega) = n, \ X_k(\omega) = X_k(\omega') \quad \forall 1 \leq k \leq n \quad \Rightarrow \quad \tau(\omega') = n. \quad (1)
\]

**Exercise 4.** Let \( \tau \) be a stopping time of a filtration \( \mathcal{F}_n \). Recall that the \( \sigma \)-algebra \( \mathcal{F}_\tau \) of “past until \( \tau \)” is defined as

\[
\mathcal{F}_\tau = \{ E : E \cap \{ \tau \leq n \} \in \mathcal{F}_n \quad \forall n \}
\]

Show that for every random variable \( V \) measurable with respect to \( \mathcal{F}_\tau \) there exists a stochastic process \( \{ G_n, n = 1, \ldots \} \), with \( G_n \) measurable with respect to \( \mathcal{F}_n \), such that

\[
V = G_\tau.
\]

(Hint: First consider simple \( V \)).

**Exercise 5.** *(Cover time of \( C_n \))* For a MC with state space \( \mathcal{X} \) we define \( \tau_{cov} \) to be the first time that every element of \( \mathcal{X} \) was visited. The covering time \( t_{cov} = \max_{x \in \mathcal{X}} \mathbb{E}^x[\tau_{cov}] \). Consider a MC that is a simple random walk on an \( n \)-cycle: it moves with probability 1/2 to one of the neighbors each time. Show that \( t_{cov}(n) = \frac{n(n-1)}{2} \) (Lovász’93). (Hint: Let \( \tau_n \) be the first time a simple random walk on \( \mathbb{Z} \) started at 0 visits \( n \) distinct states. Relate to \( t_{cov} \) and gambler’s ruin.)

**Exercise 6.** *(Last visited vertex of \( C_n \))* Consider a simple random walk \( X_t \) on an \( n \)-cycle \( C_n \) and let \( \tau_{cov} \) be the first time that every vertex was visited. Show that given that \( X_0 = v \) the distribution of \( X_{\tau_{cov}} \) is uniform on \( \{ v \}^c \). (Hint: Notice that to have \( X_{\tau_{cov}} = k \) the random walk should visit the states \( k - 1 \) and \( k + 1 \) before \( k \).)

Fun fact: cycles and cliques are the only graphs with this property (Lovász-Winkler’93).

**Exercise 7.** Let \( B_k \) be iid with law \( \mathbb{P}[B_k = +1] = p = 1 - \mathbb{P}[B_k = -1] \).

Answer the following:

- Let \( X_n = B_n B_{n+1}, n \geq 0 \). Is it Markov? If yes, find its transition kernel.

- Let \( Y_n = \frac{1}{2}(B_n - B_{n-1}), n \geq 1 \). Is it Markov? If yes, find its transition kernel.
• Let $Z_n = |\sum_{k=1}^n B_k|$, $n \geq 1$. Is it Markov? If yes, find its transition kernel.

• If $\{V_i, i \geq 0\}$ is a Markov process with state space $\mathcal{X}$, and $E_j$ are some subsets of $\mathcal{X}$, is it true that

$$P[V_n \in E_n|V_{n-1} \in E_{n-1}, V_{n-2} \in E_{n-2}, \ldots, V_0 \in E_0] = P[V_n \in E_n|V_{n-1} \in E_{n-1}],$$

provided that $P[V_{n-1} \in E_{n-1}, \ldots, V_0 \in E_0] > 0$?

• Suppose that $P(x, y)$ is a kernel of an irreducible Markov chain. If $P(\cdot, x_1) = P(\cdot, x_2)$ show that $\pi(x_1) = \pi(x_2)$, where $\pi$ is a stationary distribution. What if the chain is not irreducible?