Readings:
(a) Notes from Lecture 1.
(b) Handout on background material on sets and real analysis (Recitation 1).

Supplementary readings:
[C], Sections 1.1-1.4.
[GS], Sections 1.1-1.3.
[W], Sections 1.0-1.5, 1.9.

Exercise 1.
(a) Let \( \mathbb{N} \) be the set of positive integers. A function \( f : \mathbb{N} \rightarrow \{0, 1\} \) is said to be periodic if there exists some \( N \) such that \( f(n + N) = f(n) \), for all \( n \in \mathbb{N} \). Show that the set of periodic functions is countable.

(b) Does the result from part (a) remain valid if we consider rational-valued periodic functions \( f : \mathbb{N} \rightarrow \mathbb{Q} \)?

Exercise 2. Let \( \{x_n\} \) and \( \{y_n\} \) be real sequences that converge to \( x \) and \( y \), respectively. Provide a formal proof of the fact that \( x_n + y_n \) converges to \( x + y \).

Exercise 3. We are given a function \( f : A \times B \rightarrow \mathbb{R} \), where \( A \) and \( B \) are nonempty sets.

(a) Assuming that the sets \( A \) and \( B \) are finite, show that

\[
\max_{x \in A} \min_{y \in B} f(x, y) \leq \min_{y \in B} \max_{x \in A} f(x, y).
\]

(b) For general nonempty sets (not necessarily finite), show that

\[
\sup_{x \in A} \inf_{y \in B} f(x, y) \leq \inf_{y \in B} \sup_{x \in A} f(x, y).
\]
Exercise 4. A probabilistic experiment involves an infinite sequence of trials. For \( k = 1, 2, \ldots \), let \( A_k \) be the event that the \( k \)th trial was a success. Write down a set-theoretic expression that describes the following event:

\[ B: \text{For every } k \text{ there exists an } \ell \text{ such that trials } k\ell \text{ and } k\ell^2 \text{ were both successes.} \]

**Note:** A “set theoretic expression” is an expression like \( \bigcup_{k>5} \bigcap_{\ell<k} A_{k+\ell} \).

Exercise 5. Let \( f_n, f, g : [0, 1] \to [0, 1] \) and \( a, b, c, d \in [0, 1] \). Derive the following set theoretic expressions:

(a) Show that \( \{ x \in [0, 1] | \sup_n f_n(x) \leq a \} = \bigcap_n \{ x \in [0, 1] | f_n(x) \leq a \} \), and use this to express \( \{ x \in [0, 1] | \sup_n f_n(x) < a \} \) as a countable combination (countable unions, countable intersections and complements) of sets of the form \( \{ x \in [0, 1] | f_n(x) \leq b \} \).

(b) Express \( \{ x \in [0, 1] | f(x) > g(x) \} \) as a countable combination of sets of the form \( \{ x \in [0, 1] | f(x) > c \} \) and \( \{ x \in [0, 1] | g(x) < d \} \).

(c) Express \( \{ x \in [0, 1] | \limsup_n f_n(x) \leq c \} \) as a countable combination of sets of the form \( \{ x \in [0, 1] | f_n(x) \leq c \} \).

(d) Express \( \{ x \in [0, 1] | \lim f_n(x) \text{ exists} \} \) as a countable combination of sets of the form \( \{ x \in [0, 1] | f_n(x) < c \}, \{ x \in [0, 1] | f_n(x) > c \}, \) etc. (Hint: think of \( \{ x \in [0, 1] | \limsup_n f_n(x) > \liminf_n f_n(x) \} \)).

Exercise 6. Optional — not to be graded.
This exercise develops an example that is meant to illustrate the following: if we work with fields instead of \( \sigma \)-fields, and if we only require finite additivity, then countable additivity will not be an automatic consequence, and the model may not correspond to any intuitive notion of probabilities.

Let \( = \mathbb{N} \) (the positive integers), and let \( \mathcal{F}_0 \) be the collection of subsets of \( \mathbb{N} \) that either have finite cardinality or their complement has finite cardinality. For any \( A \in \mathcal{F}_0 \), let \( \mathbb{P}(A) = 0 \) if \( A \) is finite, and \( \mathbb{P}(A) = 1 \) if \( A^C \) is finite.

(a) Show that \( \mathcal{F}_0 \) is a field but not a \( \sigma \)-field.

(b) Show that \( \mathbb{P} \) is finitely additive on \( \mathcal{F}_0 \); that is, if \( A, B \in \mathcal{F}_0 \), and \( A, B \) are disjoint, then \( \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) \).
(c) Show that $\mathbb{P}$ is not countably additive on $\mathcal{F}_0$; that is, construct a sequence of disjoint sets $A_i \in \mathcal{F}_0$ such that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_0$ and $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) \neq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

(d) Construct a decreasing sequence of sets $A_i \in \mathcal{F}_0$ such that $\bigcap_{i=1}^{\infty} A_i = \emptyset$ for which $\lim_{i \to \infty} \mathbb{P}(A_i) \neq 0$. 