Let \( \{Y_i; i \geq 1\} \) be the IID service times for a \((G/G/\infty)\) queue and let \( \{N(t); t > 0\} \) be the renewal process with interarrival times \( \{X_i; i \geq 1\} \). Consider the following plausibility argument for \( \lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{N(t, \omega)} Y_i(\omega) \).

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{N(t, \omega)} Y_i(\omega) = \lim_{t \to \infty} \left[ \frac{N(t, \omega)}{t} \frac{\sum_{i=1}^{N(t, \omega)} Y_i(\omega)}{N(t, \omega)} \right] \tag{1}
\]

\[
= \lim_{t \to \infty} \frac{N(t, \omega)}{t} \lim_{t \to \infty} \frac{\sum_{i=1}^{N(t, \omega)} Y_i(\omega)}{N(t, \omega)} \tag{2}
\]

\[
= \lim_{t \to \infty} \frac{N(t, \omega)}{t} \lim_{n \to \infty} \frac{\sum_{i=1}^{n} Y_i(\omega)}{n} \tag{3}
\]

\[
= \frac{1}{X} \bar{Y} \tag{4} \text{ WP1}
\]

This assumes \( X < \infty, \bar{Y} < \infty \).
To do this carefully, work from bottom up.

Let $A_1 = \{ \omega : \lim_{t \to \infty} N(t, \omega)/t = 1/X \}$. By the strong law for renewal processes $\Pr\{A_1\} = 1$.

Let $A_2 = \{ \omega : \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} Y_i(\omega) = Y \}$. By the SLLN, $\Pr\{A_2\} = 1$. Thus (3) = (4) for $\omega \in A_1 A_2$ and $\Pr\{A_1 A_2\} = 1$.

Assume $\omega \in A_2$, and $\epsilon > 0$. Then $\exists m(\epsilon, \omega)$ such that $|\frac{1}{n} \sum_{i=1}^{n} Y_i(\omega) - Y| < \epsilon$ for all $n \geq m(\epsilon, \omega)$. If $\omega \in A_1$ also, then $\lim_{t \to \infty} N(t, \omega) = \infty$, so $\exists t(\epsilon, \omega)$ such that $N(t, \omega) \geq m(\epsilon, \omega)$ for all $t \geq t(\epsilon, \omega)$.

$$|\frac{\sum_{i=1}^{N(t, \omega)} Y_i(\omega)}{N(t, \omega)} - Y| < \epsilon \text{ for all } t \geq t(\epsilon, \omega)$$

Since $\epsilon$ is arbitrary, (2) = (3) = (4) for $\omega \in A_1 A_2$.

Finally, can we interchange the limit of a product of two functions (say $f(t)g(t)$) with the product of the limits? If the two functions each have finite limits (as the functions of interest do for $\omega \in A_1 A_2$), the answer is yes, establishing (1) = (4).

To see this, assume $\lim_{t} f(t) = a$ and $\lim_{t} g(t) = b$. Then

$$f(t)g(t) - ab = (f(t)-a)(g(t)-b) + a(g(t)-b) + b(f(t)-a)$$

$$|f(t)g(t) - ab| \leq |f(t)-a||g(t)-b| + |a||g(t)-b| + |b||f(t)-a|$$

For any $\epsilon > 0$, choose $t(\epsilon)$ such that $|f(t) - a| \leq \epsilon$ for $t \geq t(\epsilon)$ and $|g(t) - b| \leq \epsilon$ for $t \geq t(\epsilon)$. Then

$$|f(t)g(t) - ab| \leq \epsilon^2 + \epsilon|a| + \epsilon|b| \text{ for } t \geq t(\epsilon).$$

Thus $\lim_{t} f(t)g(t) = \lim_{t} f(t) \lim_{t} g(t)$.
Review - Countable-state chains

Two states are in the same class if they communicate (same as for finite-state chains).

Thm: All states in the same class are recurrent or all are transient.

Pf: Assume \( j \) is recurrent; then \( \sum_n P^n_{jj} = \infty \). For any \( i \) such that \( j \leftrightarrow i \), \( P^m_{ij} > 0 \) for some \( m \) and \( P^\ell_{ji} \) for some \( \ell \). Then (recalling \( \lim_t E[N_{ii}(t)] = \sum_n P^n_{ii} \))

\[
\sum_{n=1}^\infty P^n_{ii} \geq \sum_{k=n-m-\ell}^\infty P^m_{ij} P^k_{jj} P^\ell_{jk} = \infty
\]

By the same kind of argument, if \( i \leftrightarrow j \) are recurrent, then \( \sum_{n=1}^\infty P^n_{ij} = \infty \) (so also \( \lim_t E[N^t_{ij}] = \infty \)).

If a state \( j \) is recurrent, then the recurrence time \( T_{jj} \) might or might not have a finite expectation.

Def: If \( E[T_{jj}] < \infty \), \( j \) is positive-recurrent. If \( T_{jj} \) is a rv and \( E[T_{jj}] = \infty \), then \( j \) is null-recurrent. Otherwise \( j \) is transient.

For \( p = 1/2 \), each state in each of the following is null recurrent.

\[
\begin{array}{c}
\text{1} \\
\text{0} \\
\text{3} \\
\text{2}
\end{array}
\]

\[
\begin{array}{c}
\text{1} \\
\text{0} \\
\text{2} \\
\text{3}
\end{array}
\]
Positive-recurrence and null-recurrence

Suppose \( i \leftrightarrow j \) are recurrent. Consider the renewal process of returns to \( j \) with \( X_0 = j \). Consider rewards \( R(t) = 1 \) whenever \( X(t) = i \). By the renewal-reward thm (4.4.1),

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t R(\tau) d\tau = \frac{E[R_n]}{T_{jj}} \quad \text{WP1,}
\]

where \( E[R_n] \) is the expected number of visits to \( i \) within a recurrence of \( j \). The left side is \( \lim_{t \to \infty} \frac{1}{t} N_{ji}(t) \), which is \( 1/T_{ii} \). Thus

\[
\frac{1}{T_{ii}} = \frac{E[R_n]}{T_{jj}}
\]

Since there must be a path from \( j \) to \( i \), \( E[R_n] > 0 \).

Thm: For \( i \leftrightarrow j \) recurrent, either both are positive-recurrent or both null-recurrent.

---

Steady-state for positive-recurrent chains

We define steady-state probabilities for countable-state Markov chains in the same way as for finite-state chains, namely,

**Def:** \( \{\pi_i; i \geq 0\} \) is a steady-state distribution if

\[
\pi_j \geq 0; \quad \pi_j = \sum_i \pi_i P_{ij} \quad \text{for all } j \geq 0 \quad \text{and } \sum_j \pi_j = 1
\]

**Def:** An irreducible Markov chain is a Markov chain in which all pairs of states communicate.

For finite-state chains, irreducible means recurrent. Here it can be positive-recurrent, null-recurrent, or transient.
If steady-state $\pi$ exists and if $\Pr\{X_0 = i\} = \pi_i$ for each $i$, then $p_{X_1}(j) = \sum_i \pi_i P_{ij} = \pi_j$. Iterating, $p_{X_n}(j) = \pi_j$, so steady-state is preserved. Let $\tilde{N}_j(t)$ be number of visits to $j$ in $(0, t]$ starting in steady state. Then

$$E[\tilde{N}_j(t)] = \sum_{k=1}^{\infty} \Pr\{X_k = j\} = n\pi_j$$

Awkward thing about renewals and Markov: $\tilde{N}_j(t)$ works for some things and $N_{jj}(t)$ works for others. Here is a useful hack:

$N_{ij}(t)$ is 1 for first visit to $j$ (if any) plus $N_{ij}(t) - 1$ for subsequent recurrences $j$ to $j$. Thus

$$E[N_{ij}(t)] \leq 1 + E[N_{jj}(t)]$$

$$E[\tilde{N}_j(t)] = \sum_i \pi_i E[N_{ij}(t)] \leq 1 + E[N_{jj}(t)]$$

Major theorem: For an irreducible Markov chain, the steady-state equations have a solution if and only if the states are positive-recurrent. If a solution exists, then $\pi_i = 1/T_{ii} > 0$ for all $i$.

Pf: (only if; assume $\pi$ exists, show positive-recur.) For each $j$ and $i$,

$$\pi_j = \frac{E[\tilde{N}_j(t)]}{t} \leq \frac{1}{t} + \frac{E[N_{jj}(t)]}{t} \leq \lim_{t \to \infty} \frac{E[N_{jj}(t)]}{t} = \frac{1}{T_{jj}}$$

Since $\sum_j \pi_j = 1$, some $\pi_j > 0$. Thus $\lim_{t \to \infty} E[N_{jj}(t)]/t > 0$ for that $j$, so $j$ is positive-recurrent. Thus all states are positive-recurrent. See text to show that ‘$\leq$’ above is equality.
Birth-death Markov chains

For any state $i$ and any sample path, the number of $i \to i+1$ transitions is within 1 of the number of $i+1 \to j$ transitions; in the limit as the length of the sample path $\to \infty$,

$$\pi_i p_i = \pi_{i+1} q_{i+1}; \quad \pi_{i+1} = \frac{\pi_i p_i}{q_{i+1}}$$

Letting $\rho_i = p_i/q_{i+1}$, this becomes

$$\pi_i = \pi_0 \prod_{j=0}^{i-1} \rho_j; \quad \pi_0 = \frac{1}{1 + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \rho_j}.$$  

This agrees with the steady-state equations.

This solution is a function only of $\rho_0, \rho_1, \ldots$ and doesn’t depend on size of self loops.

The expression for $\pi_0$ converges (making the chain positive recurrent) (essentially) if the $\rho_i$ are asymptotically less than 1.

Methodology: We could check renewal results carefully to see if finding $\pi_i$ by up/down counting is justified. Using the major theorem is easier.

Birth-death chains are particularly useful in queuing where births are arrivals and deaths departures.
Reversibility

\[ \Pr\{X_{n+k}, \ldots X_{n+1} | X_n, \ldots X_0\} = \Pr\{X_{n+k}, \ldots, X_{n+1} | X_n\} \]

For any \( A^+ \) defined on \( X_{n+1} \) up and \( A^- \) defined on \( X_{n-1} \) down,

\[ \Pr\{A^+ | X_n, A^-\} = \Pr\{A^+ | X_n\} \]

\[ \Pr\{A^+, A^- | X_n\} = \Pr\{A^+ | X_n\} \Pr\{A^- | X_n\} . \]

\[ \Pr\{A^- | X_n, A^+\} = \Pr\{A^- | X_n\} . \]

\[ \Pr\{X_{n-1} | X_n, X_{n+1}, \ldots, X_{n+k}\} = \Pr\{X_{n-1} | X_n\} . \]

By Bayes,

\[ \Pr\{X_{n-1} | X_n\} = \frac{\Pr\{X_n | X_{n-1}\} \Pr\{X_{n-1}\}}{\Pr\{X_n\}} . \]

If the forward chain is in steady state, then

\[ \Pr\{X_{n-1} = j | X_n = i\} = P_{ji} \pi_j / \pi_i . \]

Aside from the homogeniety involved in starting at time 0, this says that a Markov chain run backwards is still Markov. If we think of the chain as starting in steady state at time \(-\infty\), these are the equations of a (homogeneous) Markov chain. Denoting \( \Pr\{X_{n-1} = j | X_n = i\} \) as the backward transition probabilities \( P^*_ji \), forward/ backward are related by

\[ \pi_i P^*_ij = \pi_j P_{ji} . \]

Def: A chain is reversible if \( P^*_ij = P_{ij} \) for all \( i,j \).
Thm: A birth/death Markov chain is reversible if it has a steady-state distribution.