Theorem: If \( \{Z_n; n \geq 1\} \) converges to \( \alpha \) WP1, (i.e., \( \Pr\{\omega: \lim_n (Z_n(\omega) - \alpha) = 0\} = 1 \)), and \( f(x) \) is continuous at \( \alpha \). Then \( \Pr\{\omega: \lim_n f(Z_n(\omega)) = \alpha\} = 1 \).

For a renewal process with inter-renewals \( X_i, 0 < \bar{X} < \infty \), \( \Pr\{\omega: \lim_n \frac{1}{n} S_n(\omega) - \bar{X} = 0\} = 1 \)

For renewal processes, \( n/S_n \) and \( N(t)/t \) are related by
The strong law for renewal processes follows from this relation between $n/S_n$ and $N(t)/t$.

Theorem: For a renewal process with $X < \infty$,

$$\Pr\left\{ \omega : \lim_{t \to \infty} N(t, \omega)/t = 1/X \right\} = 1.$$  

This says that the rate of renewals over the infinite time horizon (i.e., $\lim_{t} N(t)/t$) is $1/X$. WP1.

This also implies the weak law for renewals,

$$\lim_{t \to \infty} \Pr\left\{ \left| \frac{N(t)}{t} - \frac{1}{X} \right| > \epsilon \right\} = 0 \quad \text{for all } \epsilon > 0$$

Review of residual life

Def: The residual life $Y(t)$ of a renewal process at time $t$ is the remaining time until the next renewal, i.e., $Y(t) = S_{N(t)+1} - t$.

Residual life is a random process; for each sample point $\omega$, $Y(t, \omega)$ is a sample function.

$$\sum_{n=1}^{N(t, \omega)} \frac{X_n^2(\omega)}{2t} \leq \frac{1}{t} \int_0^t Y(t, \omega) \, dt \leq \sum_{n=1}^{N(t, \omega)+1} \frac{X_n^2(\omega)}{2t}$$
\[ \frac{N(t,\omega)}{2t} \sum_{i=1}^{N(t,\omega)} \frac{X_i^2(\omega)}{2t} \leq \frac{1}{t} \int_0^t Y(t,\omega) \, dt \leq \frac{N(t,\omega)+1}{2t} \sum_{i=1}^{N(t,\omega)} \frac{X_i^2(\omega)}{2t} \]

**Going to the limit** \( t \to \infty \)

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t Y(t,\omega) \, dt = \lim_{t \to \infty} \frac{N(t,\omega)}{2N(t,\omega)} \sum_{n=1}^{N(t,\omega)} \frac{X_n^2(\omega)}{t} = \frac{E[X^2]}{2E[X]} \]

This is infinite if \( E[X^2] = \infty \). Think of example where \( p_X(\epsilon) = 1 - \epsilon, \ p_X(1/\epsilon) = \epsilon \).

**Similar examples:** Age \( Z(t) = t - S_{N(t)} \) and duration, \( \tilde{X}(t) = S_{N(t)+1} - S_{N(t)} \).

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \tilde{X}(\tau) \, d\tau = \frac{E[X^2]}{2E[X]} \quad \text{WP1.} \]

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \tilde{X}(\tau) \, d\tau = \frac{E[X^2]}{E[X]} \quad \text{WP1.} \]
Time-averages for renewal rewards

Residual life, age, and duration are examples of assigning rewards to renewal processes.

The reward $R(t)$ at any time $t$ is restricted to be a function of the inter-renewal period containing $t$.

In simplest form, $R(t)$ is restricted to be a function $\mathcal{R}(Z(t), \bar{X}(t))$.

The time-average for a sample path of $R(t)$ is found by analogy to residual life. Start with the $n$th inter-renewal interval.

$$ R_n(\omega) = \int_{S_{n-1}(\omega)}^{S_n(\omega)} R(t, \omega) dt $$

Interval 1 goes from 0 to $S_1$, with $Z(t) = t$. For interval $n, Z(t) = t - S_{n-1}, i.e., S_N(t) = S_{n-1}$.

$$ R_n = \int_{S_{n-1}}^{S_n} R(t) dt $$

$$ = \int_{S_{n-1}}^{S_n} \mathcal{R}(Z(t), \bar{X}(t)) dt $$

$$ = \int_{S_{n-1}}^{S_n} \mathcal{R}(t - S_{n-1}, X_n) dt $$

$$ = \int_{X_n} \mathcal{R}(z, X_n) dz $$

This is a function only of the rv $X_n$. Thus

$$ E[R_n] = \int_{x=0}^{\infty} \int_{z=0}^{x} \mathcal{R}(z, x) dz dF_X(x) $$

Assuming that this expectation exists,

$$ \lim_{t \to \infty} \frac{1}{t} \int_{\tau=0}^{t} R(\tau) d\tau = \frac{E[R_n]}{X} \quad \text{WP1} $$
Example: Suppose we want to find the $k$th moment of the age.

Then $\mathcal{R}(Z(t), \tilde{X}(t)) = Z^k(t)$. Thus

$$E[R_n] = \int_{x=0}^{\infty} \int_{z=0}^{x} z^k \, dz \, dF_X(x)$$

$$= \int_{0}^{\infty} \frac{x^{k+1}}{k+1} \, dF_X(x) = \frac{1}{k} E[X^{k+1}]$$

$$\lim_{t \to \infty} \frac{1}{t} \int_{\tau=0}^{t} R(\tau) \, d\tau = \frac{E[X^{k+1}]}{(k+1)X} \quad \text{WP1}$$

**Stopping trials for stochastic processes**

It is often important to analyze the initial segment of a stochastic process, but rather than investigating the interval $(0, t]$ for a fixed $t$, we want to investigate $(0, t]$, where $t$ is selected by the sample path up until $t$.

It is somewhat tricky to formalize this, since $t$ becomes a rv which is a function of $\{X(t); \tau \leq t\}$. This approach seems circular, so we have to be careful.

We consider only discrete-time processes $\{X_i; i \geq 1\}$. 
Let $J$ be a positive integer rv that describes when a sequence $X_1, X_2, \ldots$, is to be stopped.

At trial 1, $X_1(\omega)$ is observed and a decision is made, based on $X_1(\omega)$, whether or not to stop. If we stop, $J(\omega) = 1$.

At trial 2 (if $J(\omega) \neq 1$), $X_2(\omega)$ is observed and a decision is made, based on $X_1(\omega), X_2(\omega)$, whether or not to stop. If we stop, $J(\omega) = 2$.

At trial 3 (if $J(\omega) \neq 1, 2$), $X_3(\omega)$ is observed and a decision is made, based on $X_1(\omega), X_2(\omega), X_3(\omega)$, whether or not to stop. If we stop, $J(\omega) = 3$, etc.

At each trial $n$ (if stopping has not yet occurred), $X_n$ is observed and a decision (based on $X_1 \ldots, X_n$) is made; if we stop, then $J(\omega) = n$.

Def: A stopping trial (or stopping time) $J$ for $\{X_n; n \geq 1\}$, is a positive integer-valued rv such that for each $n \geq 1$, the indicator rv $I_{\{J=n\}}$ is a function of $\{X_1, X_2, \ldots, X_n\}$.

A possibly defective stopping trial is the same except that $J$ might be defective.

We visualize ‘conducting’ successive trials $X_1, X_2, \ldots$, until some $n$ at which the event $\{J = n\}$ occurs; further trials then cease. It is simpler conceptually to visualize stopping the observation of trials after the stopping trial, but continuing to conduct trials.

Since $J$ is a (possibly defective) rv, the events $\{J = 1\}, \{J = 2\}, \ldots$ are disjoint.
Example 1: A gambler goes to a casino and gambles until broke.

Example 2: Flip a coin until 10 successive heads appear.

Example 3: Test an hypothesis with repeated trials until one or the other hypothesis is sufficiently probable a posteriori.

Example 4: Observe successive renewals in a renewal process until $S_n \geq 100$.

Suppose the rv's $X_i$ in a process $\{X_n; n \geq 1\}$ have a finite number of possible sample values. Then any (possibly defective) stopping trial $J$ can be represented as a rooted tree where the trial at which each sample path stops is represented by a terminal node.

Example: $X$ is binary and stopping occurs when the pattern $(1, 0)$ first occurs.
Wald’s equality

Theorem (Wald’s equality) Let \( \{X_n; n \geq 1\} \) be a sequence of IID rv’s, each of mean \( \bar{X} \). If \( J \) is a stopping trial for \( \{X_n; n \geq 1\} \) and if \( E[J] < \infty \), then the sum \( S_J = X_1 + X_2 + \cdots + X_J \) at the stopping trial \( J \) satisfies

\[
E[S_J] = \bar{X}E[J]
\]

Prf:

\[
S_J = X_1\mathbb{I}_{J \geq 1} + X_2\mathbb{I}_{J \geq 1} + \cdots + X_n\mathbb{I}_{J \geq n} + \cdots
\]

\[
E[S_J] = E\left[ \sum_n X_n\mathbb{I}_{J \geq n} \right] = \sum_n E\left[ X_n\mathbb{I}_{J \geq n} \right]
\]

The essence of the proof is to show that \( X_n \) and \( \mathbb{I}_{J \geq n} \) are independent.

To show that \( X_n \) and \( \mathbb{I}_{J \geq n} \) are independent, note that \( \mathbb{I}_{J \geq n} = 1 - \mathbb{I}_{J < n} \). Also \( \mathbb{I}_{J < n} \) is a function of \( X_1, \ldots, X_{n-1} \). Since the \( X_i \) are IID, \( X_n \) is independent of \( X_1, \ldots, X_{n-1} \), and thus \( \mathbb{I}_{J < n} \), and thus of \( \mathbb{I}_{J \geq n} \).

This is surprising, since \( X_n \) is certainly not independent of \( \mathbb{I}_{J=n} \), nor of \( \mathbb{I}_{J=n+1} \), etc.

The resolution of this ‘paradox’ is that, given that \( J \geq n \) (i.e., that stopping has not occurred before trial \( n \)), the trial at which stopping occurs depends on \( X_n \), but whether or not \( J \geq n \) occurs depends only on \( X_1, \ldots, X_{n-1} \).

Now we can finish the proof.
\[ E[S_J] = \sum_n E[X_n \mathbb{I}_{J \geq n}] \]
\[ = \sum_n E[X_n] E[\mathbb{I}_{J \geq n}] \]
\[ = \mathbb{X} \sum_n E[\mathbb{I}_{J \geq n}] \]
\[ = \mathbb{X} \sum_n \Pr\{J \geq n\} = \mathbb{X} E[J] \]

In many applications, this gives us one equation in two quantities neither of which is known. Frequently, \( E[S_J] \) is easy to find and this solves for \( E[J] \).

The following example shows, among other things, why \( E[J] < \infty \) is required for Wald’s equality.

Stop when you’re ahead

Consider tossing a coin with probability of heads equal to \( p \). $1 is bet on each toss and you win on heads, lose on tails. You stop when your winnings reach $1.

If \( p > 1/2 \), your winnings (in the absence of stopping) would grow without bound, passing through 1, so \( J \) must be a rv. \( S_J = 1 \) WP1, so \( E[S_J] = 1 \). Thus, Wald says that \( E[J] = 1/\mathbb{X} = \frac{1}{2p-1} \). Let’s verify this in another way.

Note that \( J = 1 \) with probability \( p \). If \( J > 1 \), i.e., if \( S_1 = -1 \), then the only way to reach \( S_n = 1 \) is to go from \( S_1 = -1 \) to \( S_m = 0 \) for some \( m \) (requiring \( J \) steps on average); \( J \) more steps on average then gets to 1. Thus \( J = 1 + (1-p)2J = \frac{1}{2p-1} \).
Next consider \( p < 1/2 \). It is still possible to win and stop (for example, \( J = 1 \) with probability \( p \) and \( J = 3 \) with probability \( p^2(1-p) \)). It is also possible to head South forever.

Let \( \theta = \Pr\{J < \infty\} \). Note that \( \Pr\{J = 1\} = p \). Given that \( J > 1 \), i.e., that \( S_1 = -1 \), the event \( \{J < \infty\} \) requires that \( S_m - S_1 = 1 \) for some \( m \), and then \( S_n - S_m = 1 \) for some \( n > m \). Each of these are independent events of probability \( \theta \), so

\[
\theta = p + (1 - p)\theta^2
\]

There are two solutions, \( \theta = p/(1 - p) \) and \( \theta = 1 \), which is impossible. Thus \( J \) is defective and Wald’s equation is inapplicable.

Finally consider \( p = 1/2 \). In the limit as \( p \) approaches 1/2 from below, \( \Pr\{J < \infty\} = 1 \). We find other more convincing ways to see this later. However, as \( p \) approaches 1/2 from above, we see that \( E[J] = \infty \).

Wald’s equality does not hold here, since \( E[J] = \infty \), and in fact does not make sense since \( \bar{X} = 0 \).

However, you make your $1 with probability 1 in a fair game and can continue to repeat the same feat.

It takes an infinite time, however, and requires access to an infinite capital.