Problem 1

(a) Assume \( h(x) > 0 \). A consumer with \( v_i \) is better off purchasing the good if \( v_i h(x) - c \geq 0 \). Thus, the best response is \( \hat{x} = 1 - F(c/h(x)) \). Note that the map \( x \mapsto \hat{x} \) is continuous and maps a compact interval \([0, 1]\) to itself. By Brouwer’s fixed-point theorem, an equilibrium exists.

(b) Omitted.

Problem 2

(a) It describes a good that a consumer want some people to possess but not many. For example, a party venue that gets better with more attendance, but gets worse when it is too crowded. The value \( v_i \) measures how much player \( i \) likes to party. The value \( p \) is a cover charge for the club; if it is too high there is no equilibrium with positive attendance.

(b) First, we need to check end points: \( x = 0 \) is an equilibrium since \( u_i < 0 \), while \( x = 1 \) is not. Second, we check interior solutions. Note that \( x \geq 1/2 \) cannot be an equilibrium since then \( g(x) \leq 0 \). Let \( \bar{v} \) be such that consumers \([\bar{v}, 1]\) purchase and others do not. Since \( v_i \sim U[0, 1] \), we have \( x^* = 1 - \bar{v} \) in interior equilibria. Thus, \( x^* \) solves \((1 - x^*)g(x^*) - p = 0\). The solutions that satisfy \( x^* < 1/2 \) are \( \frac{1 - \sqrt{1 - 4p}}{2} < \frac{1}{4} \) and \( \frac{3 - \sqrt{1 + 16p}}{4} > \frac{1}{4} \).

(c) 0 and \( \frac{3 - \sqrt{1 + 16p}}{4} \) are stable as small deviation will induce incentives to correct it; \( \frac{1 - \sqrt{1 - 4p}}{2} \) is not since any small deviation will shift the equilibrium to either of the previous two.

(d) Suppose consumers with values higher than \( 1 - x \) purchase the good. Then the social welfare is

\[
\int_{1-x}^{1} [v g(x) - p] dv = \frac{x(2-x)}{2} g(x) - px
\]

This is maximized at \( x^* = 1/4 \). Therefore, no equilibrium attains the social optimum.
Problem 3

The circle graph with four players have the adjacency matrix of

\[ G = \begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}. \]

Recall that agent i's best response function is

\[ BR_i(x_{-i}) = \max \left\{ 0, 1 - \delta \sum_{j \neq i} g_{ij}x_j \right\}. \]

First, consider an equilibrium where everyone is active. The condition is

\[
\begin{align*}
x_1 &= 1 - \delta(x_2 + x_3) > 0, \\
x_2 &= 1 - \delta(x_1 + x_4) > 0, \\
x_3 &= 1 - \delta(x_1 + x_4) > 0, \\
x_4 &= 1 - \delta(x_2 + x_3) > 0.
\end{align*}
\]

This yields \((x_1^*, x_2^*, x_3^*, x_4^*) = \left( \frac{1}{1+\delta}, \frac{1}{1+\delta}, \frac{1}{1+\delta} \right)\) for any \(\delta \geq 0\). Second, consider an equilibrium with three active agents.

\[
\begin{align*}
x_1 &= 1 - \delta(x_2 + x_3) > 0, \\
x_2 &= 1 - \delta x_1 > 0, \\
x_3 &= 1 - \delta x_1 > 0, \\
x_4 &= 1 - \delta(x_2 + x_3) = 0,
\end{align*}
\]

which is impossible. Third, consider an equilibrium with two active agents. By the same exercise, we know it is impossible to have agents 1 and 2 active. For the case with agents 1 and 4 active, we have

\[
\begin{align*}
x_1 &= 1, \\
x_2 &= 1 - \delta(x_1 + x_4) \leq 0, \\
x_3 &= 1 - \delta(x_1 + x_4) \leq 0, \\
x_4 &= 1.
\end{align*}
\]

This yields \((x_1^*, x_2^*, x_3^*, x_4^*) = (1, 0, 0, 1)\) as long as \(\delta \geq 1/2\). Note also that its rotation \((0, 1, 1, 0)\) is also an equilibrium.

Finally, we can verify that there is no equilibrium with one or zero active agent. Thus, there are two equilibria as derived above.

Problem 4

Consider the equilibrium strategy in which player 1 plays C, D, C, D, \ldots as long as player 2 plays D, C, D, C, \ldots, and vice versa. If the opponent deviates, then each player
commits to play D forever. In period 1, player 1’s anticipated payoff along the given equilibrium path is

\[-1 + 6\delta - 1\delta^2 + 6\delta^3 - \cdots = \sum_{k=0}^{\infty} (-1 + 6\delta)\delta^{2k} = \frac{-1 + 6\delta}{1 - \delta^2}.
\]

By deviating to D, he obtains

\[0 + 0\delta + 0\delta^2 + \cdots = 0.\]

Thus, we need \(\delta \geq 1/6\). It is easy to check that player 2 in period 1 (or player 1 in period 2) has no incentive to deviate if \(\delta \geq 1/6\). Also, it is easy to see that if either has deviated (so they are in an off-path state), then there is no incentive for either to deviate from playing D forever. Hence, the given strategies constitute an equilibrium if \(\delta \geq 1/6\).

Recall that the equilibrium payoff by cooperation in every period is \(2 + 2\delta + 2\delta^2 + \cdots = \frac{2}{1-\delta}\). Since \(\frac{1+6\delta}{1-\delta^2} + \frac{6-\delta}{1-\delta} - \frac{2}{1-\delta} - \frac{2}{1-\delta} = \frac{1}{1-\delta} > 0\), we see that this alternating equilibrium earns higher welfare.

**Problem 5**

(a) The pure strategy equilibria are \((B, B)\) and \((C, C)\).

(b) In the second period, the highest payoff attainable is 1 at \((B, B)\) since it is the last period. In the first period, the highest possible payoff is 3 at \((A, A)\). I argue that payoff 3 in the first stage is attainable. Consider the strategy in which a player takes A in the first period, and depending on the opponent’s action in the first period, the player determines the second-stage action; in particular, he takes B in the second period if the opponent took A in the first period, and takes C otherwise. The pair of this strategy earns a payoff of 4, while if one deviates, one can at most obtain \(4 - 1 = 3\). Therefore, there is no incentive to deviate and hence it is a subgame perfect equilibrium. Thus, the highest welfare attainable in a SPE is \(2(3 + 1) = 8\).